2024/09/23 - Transform Techniques - Week 03

Find F.T of e^{-ax^2} , $a > 0$

otherwise F.T cannot be found

$$f(x) = e^{-ax^2}$$

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)\, e^{-ixw}\, dx$$

$$\therefore \quad (\hat{f})'(w) = \int_{-\infty}^{\infty} f(x)\, e^{-ixw}\, (-ix)\, dx$$

$$= -i \int_{-\infty}^{\infty} [x\, f(x)]\, e^{-ixw}\, dx$$

$$f'(x) = e^{-ax^2} \cdot (-2ax)$$
$$= -2ax \ f(x)$$

$$(f')^{\wedge}(\omega) = \int_{-\infty}^{\infty} f'(x)\, e^{-ix\omega}\, dx$$

$$= \int_{-\infty}^{\infty} (-2ax)\, f(x)\, e^{-ix\omega}\, dx$$

$$= -2a \int_{-\infty}^{\infty} [x f(x)]\, e^{-ix\omega}\, dx$$

$$= -2a\, i (\hat{f})'(\omega) \longrightarrow \circledast$$

$$(\hat{f'})(\omega) = \int_{-\infty}^{\infty} f'(x)\, e^{-ix\omega}\, dx$$

By parts

$$\lim_{|x|\to\infty} f(x)\, e^{-ix\omega}$$

$$\leq \lim_{|x|\to\infty} |f(x)|$$

$$e^{-ax^2}$$

$$= \cancel{f(x)\, e^{-ix\omega} \Big|_{-\infty}^{\infty}} \to 0 \quad - \int f(x)\, (-i\omega)\, e^{-ix\omega}\, dx$$

$$= i\omega \int_{-\infty}^{\infty} f(x)\, e^{-ix\omega}\, dx = i\omega\, \hat{f}(\omega) \longrightarrow (**)$$

(*) and (**):

$$-2a\cancel{i}(\hat{f})'(\omega) = \cancel{i}\omega\, \hat{f}(\omega)$$

$$\therefore \frac{(\hat{f})'(\omega)}{\hat{f}(\omega)} = \frac{\omega}{-2a}$$

$$\hat{f}(\omega) = c \cdot e^{-\omega^2/a}$$

$$\hat{f}(0) = c$$

$$c = \hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} e^{-ax^2}\, dx$$

$$= 2 \int_{0}^{\infty} e^{-ax^2}\, dx$$

$$= \frac{2}{\sqrt{a}} \int_{0}^{\infty} e^{-x^2}\, dx$$

$x^2 = t$

$dx = \frac{t}{2\sqrt{t}}$

$x^2 = t$

$$= \frac{\cancel{2}}{\sqrt{a}} \underbrace{\int_{0}^{\infty} e^{-t}\, e^{+1/2 - 1} \frac{dt}{\cancel{2}}}_{\sqrt{\pi}}$$

$$= \frac{\cancel{2}}{\sqrt{a}}$$

$\therefore \quad c = \frac{\sqrt{\pi}}{a}$

$$\therefore \quad \boxed{\hat{f}(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/a}} \longrightarrow f(x) = e^{-ax^2}$$

$a = 1$

Gaussian f^n

Uncertainity principle

$\Delta_1 = \Delta_2$

pattern in time domain = pattern in frequency domain

$$\boxed{\begin{array}{l}\text{Matlab convention:} \\ \hat{f}(\omega) = \displaystyle\int_{-\infty}^{\infty} e^{-2\pi i x \omega} f(x)\, dx \\ f(x) = \displaystyle\int_{-\infty}^{\infty} e^{2\pi i x \omega} \hat{f}(\omega)\, d\omega \end{array}}$$

Laplace Transform

some information you cannot easily analysis

Integral transform

$$\underbrace{F(w)}_{} = \int_a^b \underbrace{k(\omega, t)}_{\text{kernel}} \, f(t)\, dt$$

$$\left.\begin{aligned} a &= -\infty \\ b &= \infty \\ k &= e^{-\omega_2 t} \end{aligned}\right\} \rightarrow \text{Fourier Transform}$$

If $a = -\infty$, $b = \infty$, $k(\omega, t) = e^{-inw}$

$F(w)$ is the F.T. of $f(t)$

If $a = 0$, $b = \infty$, $k(w, t) = e^{-wt}$

$\rightarrow$ $F(w)$ is the L.T of $f(t)$

$$F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

depending on f
$F(s)$ is finite/meaningful
only for some s

Exponentially bounded function

Suppose f is s.t. $|f(t)| \leq M \cdot e^{\alpha t}$ $\forall t$, $M \geq 0$

α is a fixed no.

i.e., f should not grow like $c_1 \cdot e^{c_2 t^2}$

Note: f may or may not be bounded in $\mathbb{R}$.

$$|F(s)| \leq \int_0^\infty e^{-st} \, |f(t)| \, dt \leq M \cdot \int_0^\infty \underbrace{e^{-st} \cdot e^{\alpha t}}_{e^{-(s-\alpha)t}} dt$$

$$= M \cdot \left(\frac{e^{-(s-\alpha)\cdot t}}{-(s-\alpha)} \Big|_0^\infty \right)$$

$$= M \cdot \frac{1}{s-\alpha} \longrightarrow \text{meaningful for } s > \alpha$$

$F(s) \rightarrow$ well defined for $s > \alpha$

<u>Properties</u>:

* L.T is linear

$$LT(af + bg) = a\, LT(f) + b\, LT(g) \quad \forall\, f, g$$

* ***

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\lim_{s \to \infty} F(s) = \int_0^\infty \lim_{s \to \infty} e^{-st} f(t)\, dt = 0$$

limit of integral

integral of limit

$\rightarrow$ may not be same, same in this case

e.g.: find inverse laplace transform of

$$F(s) = \frac{s^3}{s^2+1}$$

Sol^n: does not exist

$$\because \lim_{s \to \infty} F(s) \neq 0$$

* If $F(s) = LT(f(t))$, then

$$f(t) = LT^{-1}(F(s))$$

* $\underbrace{LT(f(t))}_{F(s)} = \underbrace{LT(g(t))}_{G(s)}$

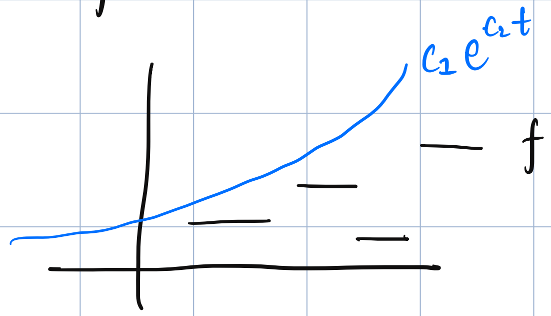
If $F(s) = G(s)$, is $f(t) = g(t) \quad \forall t$?

$$\underbrace{\int_0^\infty e^{-st}(f(t) - g(t))\,dt}_{\downarrow} = 0$$

f and g can differ over countable number of points

$f(t) = g(t) \quad \forall t$ except over a countable set

Note: f can be discontinuous



last lec notes

2024/09/27

Find L.T of

1) $f(t) = t^n$
2) $f(t) = e^{\alpha t}$
3) $f(t) = \cos \alpha t$
4) $f(t) = \sin \alpha t$
5) $f(t) = 1/\sqrt{t}$, $t > 0$
6) $f(t) = \cosh(at)$
7) $f(t) = \sinh(at)$
8) If $F(s) = \underset{\text{of } f(t)}{LT}$ find LT of $f(\alpha t)$

1) $F(s) = \int e^{-st} \cdot t^n \, dt$

$\int$ by parts

$$t^n \int e^{-st} + \frac{1}{s}\int n t^{n-1} e^{-st} \, dt$$

$$\cancel{\frac{t^n}{-s} e^{-st}} + \frac{n}{s} \int e^{-st} t^{n-1} \, dt \quad \Big|$$

$$F_n(s) = \frac{n}{s} F_{n-1}(s)$$

F_1

Answer key

① $F(s) = \dfrac{n!}{s^{n+1}}$

② $F(s) = \dfrac{1}{s-\alpha}$, $s > \alpha$

③ $F(s) = \dfrac{s}{s^2+\alpha^2} \longrightarrow$ ~~$s \notin (-\infty, \infty)$~~ s

$\boxed{s > 0}$ why?

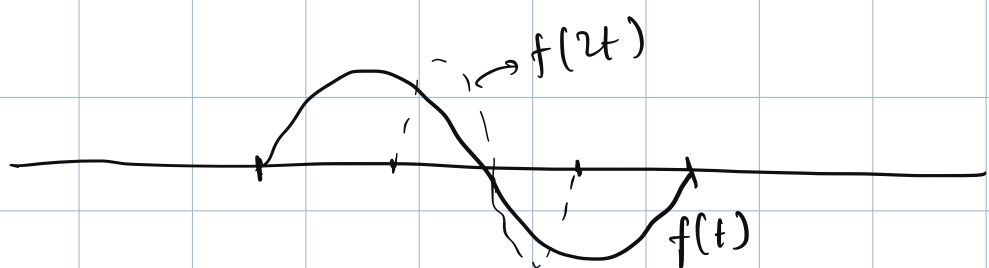
④ $F(s) = \dfrac{\alpha}{s^2+\alpha^2}$, $s > 0$

⑤ $F(s) = \sqrt{\dfrac{\pi}{s}}$, $s > 0$

⑥ $F(s) = \dfrac{s}{s^2-\alpha^2}$, $|s| > |\alpha|$

⑦ $F(s) = \dfrac{\alpha}{s^2-\alpha^2}$, $|s| > |\alpha|$

⑧



$$\mathcal{F}(f(at)) = \frac{1}{a} F\left(\frac{1}{a}\right)$$

compress in one domain $\Longrightarrow$ elongation in other domain

Laplace Transform of differentiable functions

Suppose f satisfies $|f(t)| \leq M \cdot e^{\alpha t}$ $\forall t$, $M > 0$, α is a fixed no.

and is differentiable and f' is continuous

$$L.T.(f'(t)) = s\,F(s) - f(0)$$

Corollary

$$L.T.\left(f^{(n)}(t) \right) = s^n F(s) - s^{n-1} f(0) \quad \dots \quad f^{(n-1)}(0)$$

Proof:

$$L.T.\left(f'(t)\right) = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= e^{-st} f(t) + s \cdot \underbrace{\int e^{-st} f(t)\, dt}_{F(s)} \Big|_0^\infty$$

$$= -f(0) + s \cdot F(s)$$

- To do: corollary proof.

Inverse Laplace transform

If $\quad LT(f(t)) = F(s)$

$$f(t) = \underbrace{L.T^{-1}}_{\text{inverse transform}}(F(s))$$

Examples:

Find ILT (inverse laplace transform) of

1) $F(s) = \dfrac{1}{s(s+1)}$

$$LT(e^{\alpha t}) = \frac{1}{s+\alpha}$$

$\alpha = 0$

$\Rightarrow \dfrac{1}{s}$

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$ILT\left(\frac{1}{s(s+1)}\right) = ILT\left(\frac{1}{s}\right) - ILT\left(\frac{1}{s+1}\right)$$

2) $\dfrac{s}{s^2+1}$

3) $\cos^2 s$

More problems :

1) $LT\left(t \cdot f(t)\right) = -F'(s)$ where $F(s) = LT\left(f(t)\right)$

2) $LT\left(\dfrac{f(t)}{t}\right) = \displaystyle\int_s^\infty F(p)\, dp$

f is sufficiently well bounded

$*$ $LT\left(t \cdot f(t)\right) = \displaystyle\int_0^\infty \underline{e^{-st}}\; t \cdot \underline{f(t)}\; dt$

$\cancel{= t \int e^{-st} f(t) - \int \int e^{-st} f(t)\, dt}$

$\cancel{=}$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$-F'(s) = \int_0^\infty (+t) \cdot e^{-st} f(t)\, dt$$

$$= LT\left(f(t)\right)$$

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let $\quad g(t) = \dfrac{f(t)}{t}$

$$\Rightarrow f(t) = t \cdot g(t)$$

$$F(s) = LT\left(t \cdot g(t)\right) = -G'(s)$$

where $\quad G(s) = -\int_a^s F(p)\, dp$

But $\lim_{s\to\infty} G(s) = 0 \implies \int_a^\infty F(p)\,dp = 0$

$$\therefore \quad G(s) = 0 - \int_a^s F(p)\,dp = \int_a^\infty F(p)\,dp - \int_a^s F(p)\,dp$$

$$= \int_s^\infty F(p)\,dp$$

2024/10/01

Theorem:

First limit theorem : $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} F(s)$ $\quad [\mathcal{L}(f(t)),\ \mathcal{L}(f'(t))$ exist $]$

Second limit theorem $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot F(s)$

$$\mathcal{L}(f'(t)) = s \cdot F(s) - f(0)$$

$$\lim_{s \to \infty} \int_0^\infty f'(t)\, e^{-st} = \lim_{s \to \infty} \left(s F(s) - \lim_{t \to 0^+} f(t) \right)$$

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s \cdot F(s)$$

o Proof of second

Unit step function

$$u_a(t) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \geq a \end{cases}$$

$$\mathcal{L}(u_a(t)) = \int_a^{\infty} e^{-st}\, dt$$

$$= \frac{e^{-st}}{s}\Big|_{\infty}^{a}$$

$$= \frac{e^{-as}}{s}$$

If $g(t) = \begin{cases} 0 & 0 \le t < a \\ f(t-a) & \text{if } t \geq a \end{cases}$

and $\mathcal{L}(f(t)) = F(s)$

Find $G(s)$.

$$G(s) = \int_a^{\infty} e^{-st} f(t-a)\, dt$$

$$= e^{-as} \cdot F(s)$$

Problems :

① Find $\mathcal{L}^{-1}\left(\dfrac{e^{-5s}}{s^2-4s-5}\right)$

② Find $\mathcal{L}^{-1}\left(\dfrac{(s+1)(s+3)}{s(s+2)(s+8)}\right)$

③ Find $\mathcal{L}^{-1}\left(\dfrac{3s+4}{(s+1)(s^2+4s+4)}\right)$

④ Find $F(s)$, where $f(t) = \begin{cases} 0 & t < 2\pi \\ t\cos t & t \geq 2\pi \end{cases}$

① $F(s) = \dfrac{e^{-5s}}{s^2-4s-5} = \dfrac{e^{-5s}}{s^2-5s+s-5} = \dfrac{e^{-5s}}{s(s-5)+(s-5)}$

$$= \frac{e^{-5s}}{(s+1)(s-5)}$$

$$\frac{e^{-5s}}{(s+1)(s-5)} = e^{-5s}\left(\frac{A}{s+1} + \frac{B}{s-5}\right)$$

$$A + B = 0$$

$$-5A + B = 1$$

$$(A = -1/6)$$

$$\frac{e^{-5s}}{(s+1)(s-5)} = e^{-5s}\left(\frac{(-1/6)}{(s+1)} + \frac{1/6}{(s-5)}\right)$$

$$G(s) = e^{-as} F(s)$$

$$LT\left(e^{at} f(t)\right) = F(s-a)$$