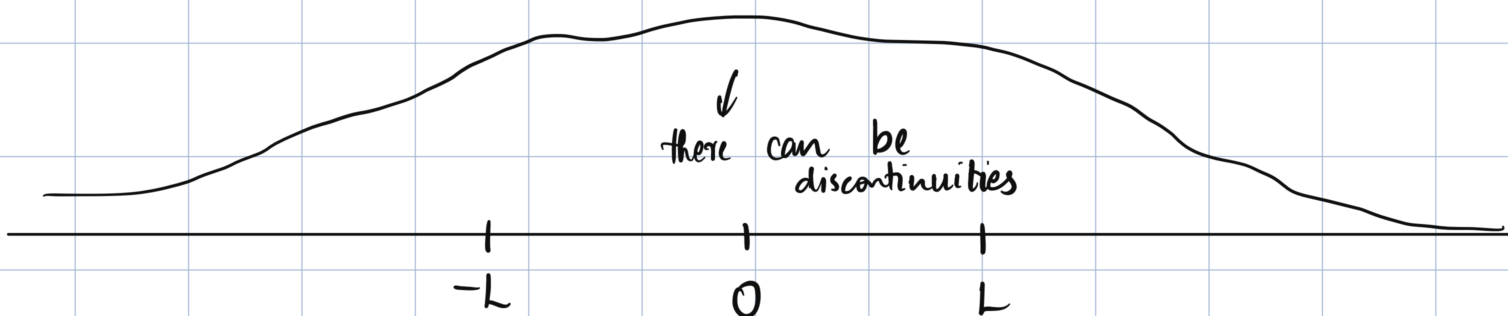
2024/09/17 - Transform Techniques - Week 02

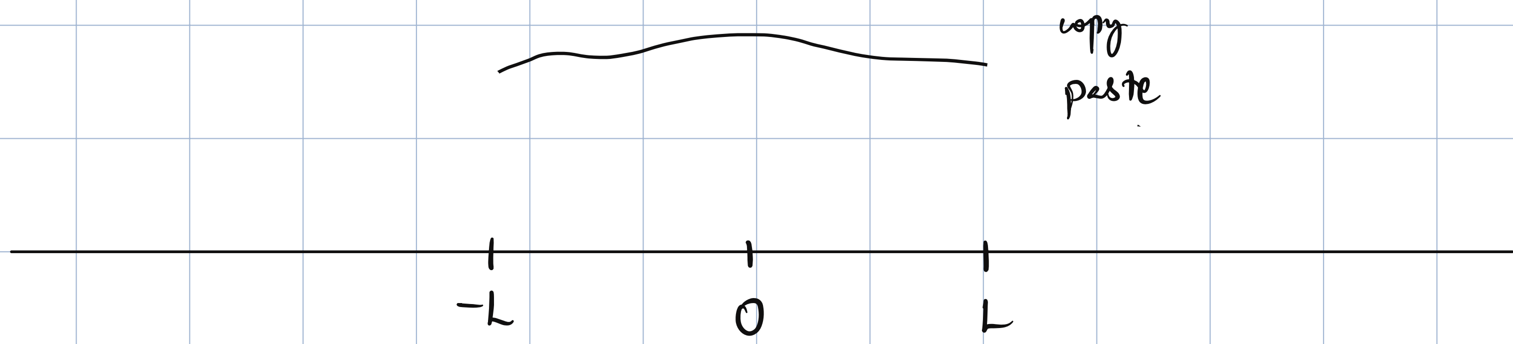
Fourier transform → generalisation of Fourier series

for any function

for periodic function

f is defined in $(-\infty, \infty)$





$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = f(x) \quad \forall \quad x \in (-L, L)$$

$$\boxed{\lim_{L\to\infty} g(x) = f(x \quad \forall \quad x \in \mathbb{R}}$$

→ not mathematically correct implication
(series might not converge)
assume f is nice
well-behaved
dirichlet

"Cosine" term

$$= \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt\right)\cdot\cos\left(\frac{n\pi x}{L}\right)$$

$$\Delta\alpha_n = \alpha_{n+1}-\alpha_n$$
$$= \pi/L$$

$$f(x) = \cancel{\lim_{L\to\infty} g(x)} = \lim_{L\to\infty}\ \lim_{N\to\infty}\sum_{N=1}^{N}\frac{1}{\pi}\underbrace{\left(\int_{-L}^{L} f(t)\cos(\alpha_n t)\,dt\right)}_{L_n(t)}\underbrace{\cos\alpha_n x\cdot\Delta\alpha_n}_{\Phi_n(x)}$$

$\phi_n(x)$

$\uparrow$

$f(\alpha)$

$$= \lim_{N\to\infty}\ \lim_{L\to\infty}\sum_{N=1}^{\infty} L_n(t)\cdot\Phi_n(x)\,\Delta\alpha_n$$

$\int_0^N h_n(t)\,\Phi_n(x)\,d\alpha$

$$= \int_0^{\infty}\left(\frac{1}{\pi}\left(\int_{-\infty}^{\infty} f(t)\cos(\alpha t)\,dt\right)\cos\alpha x\ d\alpha\right.$$

"Sine" term as $L \to \infty$ $= \int_0^\infty \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \alpha t \, dt \right) \sin \alpha x \, d\alpha$

$$|a_0| = \left| \frac{1}{L} \int_{-L}^{L} f(x)\, dx \right| \leq \frac{1}{L} \int_{-L}^{L} |f(x)|\, dx \leq \underbrace{\frac{1}{L} \int_{-\infty}^{\infty} |f(x)|\, dx}$$

suppose integral is finite then as $L \to \infty$ term tends to 0

$$f(x) = \int_0^\infty \left[\frac{1}{\pi} \left(\int_{-L}^{L} \cos \alpha t \cdot f(t)\, dt \right) \cos \alpha x + \left(\frac{1}{\pi} \int_{-L}^{L} \sin \alpha t \cdot f(t)\, dt \right) \sin \alpha x \right] d\alpha$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha t} f(t)\, dt \right) \cdot e^{i x \alpha}\, d\alpha$$

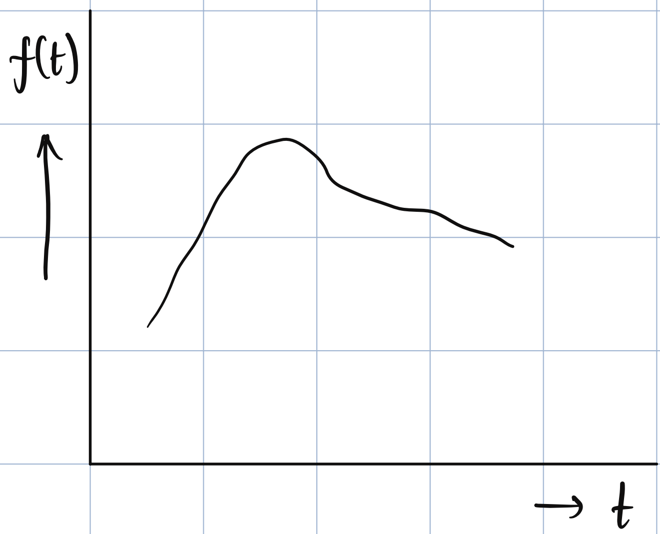
frequency

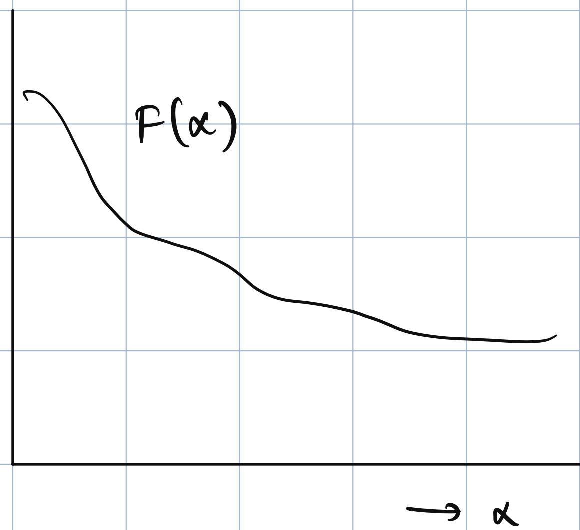
$$\hat{f}(\alpha) = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\alpha t}\, dt \longrightarrow \text{forward fourier transform}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underset{\hat{f}(\alpha)}{\underset{\downarrow}{F(\alpha)}}\, e^{i\alpha x}\, d\alpha \longrightarrow \text{inverse fourier transform}$$

Actual derivation of fourier series expansion → 60 pages

Shanon ⟶ Bell labs





$\alpha \uparrow$

term sin oscillates faster

Time - Frequency analysis

Theorem: f is a function that is piecewise continuous over every bounded interval in $(-\infty, \infty)$ and

finite energy $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$, then the fourier integral

$$\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\left[\cos \alpha t \cdot \cos \alpha x + \sin \alpha t \cdot \sin \alpha x\right) d\alpha \; dt$$ converges

to $\begin{cases} f(x) & \text{when } f \text{ is continuous at } x \\ \dfrac{f(x+) + f(x-)}{2} & \text{when } f \text{ is discontinuous at } x. \end{cases}$

data $\rightsquigarrow$ in practicality (over finite interval) $\longrightarrow$ 0 outside

$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

$f(x) = x^2$

area not defined

Suppose $f = \sin x$

If f is even function,

$$f(x) = f(-x)$$

$$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx$$

$\longrightarrow$ Fourier cosine transform

If f is odd, i.e., $f(-x) = -f(x)$

$$\hat{f}(\alpha) = \sqrt{2/\pi} \int_0^\infty f(x) \cdot \sin \alpha x \, dx$$

$\longrightarrow$ Fourier sine transform

Shannon's Sampling Theorem

Suppose f is such that $\underbrace{\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty}_{\text{square integrable}}$

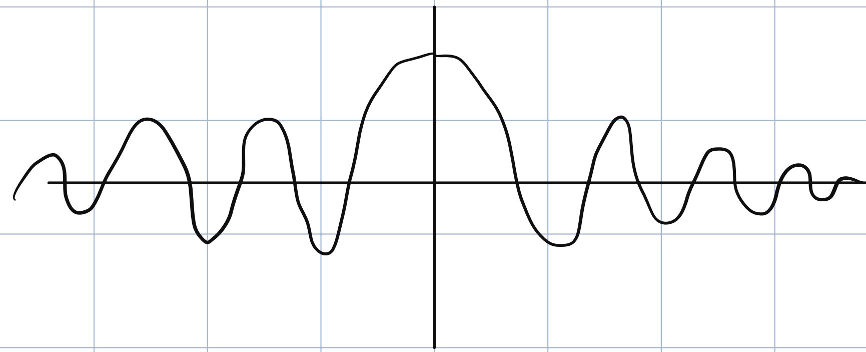
and

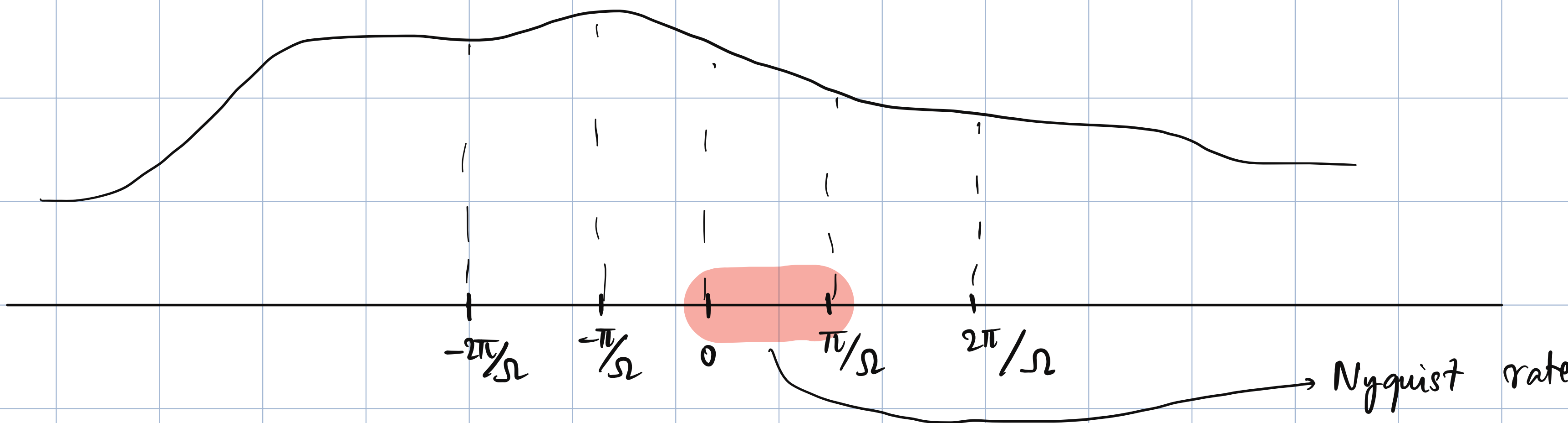
$$\hat{f}(\omega) = 0 \quad \forall \; \omega \notin [-\Omega, \Omega]$$

$$\Omega > 0$$

then

$$f(x) = \sum_{n} f\left(\frac{n\pi}{\Omega}\right) \underbrace{\frac{\sin(\Omega x - n\pi)}{\Omega x - n\pi}}_{\text{sinc function}}$$





$$f(x) \longleftrightarrow \left\{ f\left(\frac{n\pi}{\Omega}\right) \right\}$$

analog data or continuous time data

can be recovered from

digital or discrete time data

- Television signal
- connects discrete and continuous worlds
- frequency limited / band limited function

f active in some $[-\Omega, \Omega]$

if not, there are results taught in a signal processing or image processing course

Problem

① Find F.T. of $\begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$

Before apply F.T blindly see if the function satisfies the required condition or not

② Find F.T. of $e^{-\alpha x^2}$

identical

$\hat{f} = f$

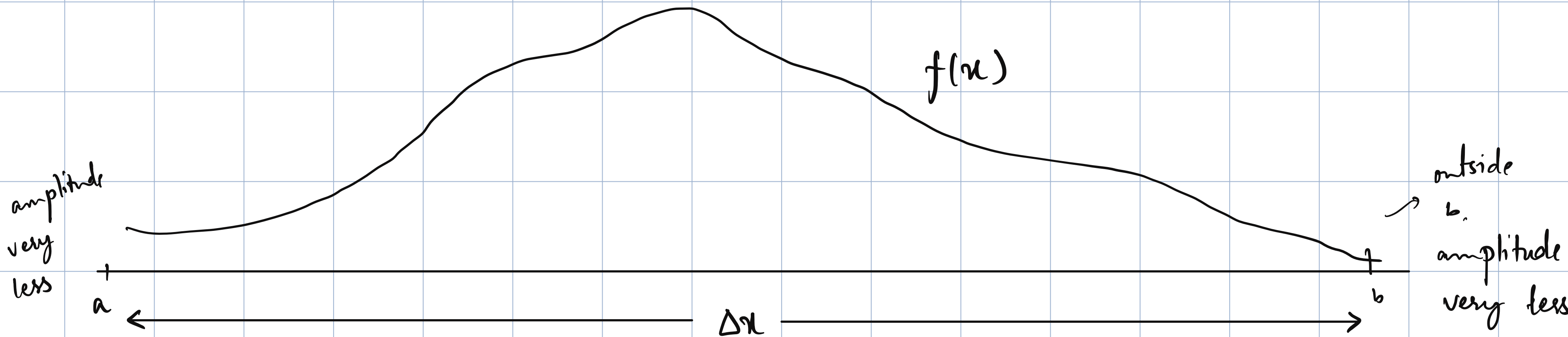
Uncertainity principle

$$\Delta x = \int_{-\infty}^{\infty} |x f(x)| \, dx$$

$$\Delta \omega = \int_{-\infty}^{\infty} |\omega \hat{f}(\omega)| \, d\omega$$

$$\Delta x \cdot \Delta \omega \geq 1/2$$

$\downarrow \Rightarrow \Delta\omega \uparrow$



f active in [a, b]

$\Delta x \rightarrow$ width of f^n in time domain

$\Delta \omega \rightarrow$ width of f^n in frequency domain

You cannot localise the function in both domains

2024|09|20

Recall :

f is " nice " function

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-it\omega}\, dt$$

$-2\pi j t \omega$ (convention)

$$\boxed{j = \frac{i}{2\pi}}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{it\omega}\, d$$

$2\pi j t \omega$

Shanon Sampling Theorem

f is s.t.

① $\hat{f}(\omega) = 0 \quad \forall \quad \omega \notin [-\Omega, \Omega] \quad , \quad \Omega > 0$

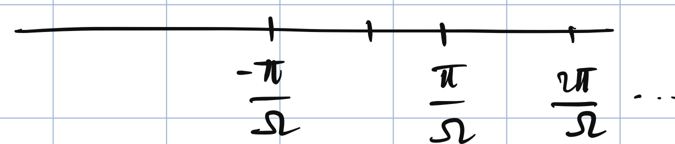
② $\underbrace{\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega \quad < \infty}_{\text{square integrable}}$

$$f(x) = \sum_n f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega x - n\pi)}{\Omega x - n\pi}$$

$f(x) \longleftrightarrow f\left(\frac{n\pi}{\Omega}\right)$

$f(x)$ ↓ continuous data / analog data

$f\left(\frac{n\pi}{\Omega}\right)$ ↓ discrete data / digital data



$$g(\omega) = \sum c_n e^{inx}$$

$$c_n = f\left(\frac{n\pi}{\Omega}\right)$$

~ periodic extension ~

??

$$f(x) = \int_{-\infty}^{\infty} \hat{g}(\omega) \, e^{i\omega x} \, dx$$

0 outside $[-\Omega, \Omega]$

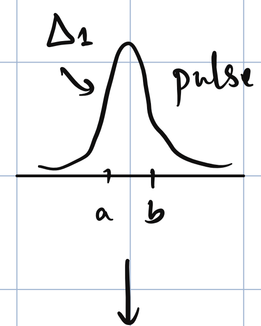
$=$

Uncertainity Principle

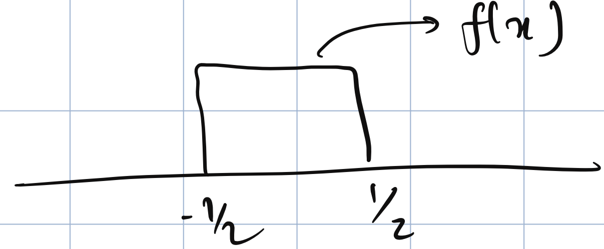
$$\Delta_1 = \int_{-\infty}^{\infty} |x f(x)|^2 \, dx$$

fourier domain

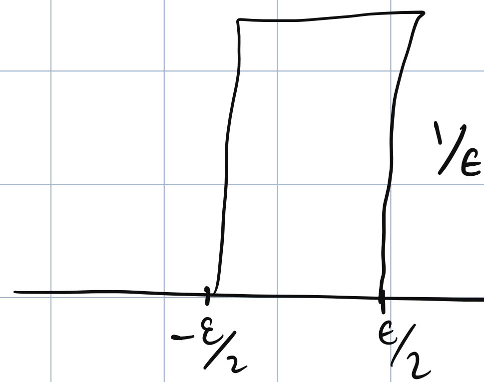
$$\Delta_2 = \int_{-\infty}^{\infty} |\omega \hat{f}(\omega)|^2 \, d\omega$$



$$g_\epsilon(x) = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)$$



$$\int_{-\epsilon/2}^{\epsilon/2} \underbrace{g_\epsilon(x)}_{1/\epsilon} \, e^{-ix\omega} \, dx$$



$$\downarrow$$

$$\frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} \cos x\omega \, dx = \left. \frac{-\sin x\omega}{\epsilon\omega} \right|_{-\epsilon/2}^{\epsilon/2}$$

Convolution

f, g are square integrable

$$h(x) = \int_{-\infty}^{\infty} f(y) \cdot g(x-y)\, dy$$

$$\downarrow$$

$$f * g\,(x)$$

$$\widehat{(f*g)}(\omega) = \hat{h}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$$

$$\hookrightarrow = \int_{-\infty}^{\infty} h(x)\, e^{-ix\omega}\, dx$$

$$\int_{-\infty}^{\infty} h(x)\, e^{-ix\omega}\, dx \quad (x)$$

$$= \int_{-\infty}^{\infty}_{(x)} \left(\int_{-\infty}^{\infty}_{(y)} f(y) \cdot g(x-y)\, dy \right) e^{-ix\omega}\, dx$$

$$= \underbrace{\left(\int_{-\infty}^{\infty} f(y) \cdot e^{-iy\omega}\, dy \right)}_{\hat{f}(\omega)} \cdot \underbrace{\left(\int_{-\infty}^{\infty} g(x)\, e^{-ix\omega}\, dx \right)}_{\hat{g}(\omega)}$$

$$\widehat{(f * g)}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$$

Why convolution? $\longrightarrow$ read

$\rightarrow$ Image processing, choose g such that it minimises error

$$\underbrace{\int_{-\infty}^{\infty} |f(x)|^2 \, dx}_{\text{Energy in } x\text{-domain}} = \underbrace{\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega}_{\text{Energy in } \omega\text{-domain (frequency)}}$$

Periodic case

$$\int_{-L}^{L} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_n (a_n^2 + b_n^2)$$

Proof: Take $g(t) = f(-t) \implies g(-t) = \overline{f(t)}$

$$f * g(x) = \int_{-\infty}^{\infty} f(y) \, g(x-y) \, dy$$

Set $x = 0$

$\bar{f}$ is the complex conjugate of f

$$f * g\,(0) = \int_{-\infty}^{\infty} f(y) \cdot \underbrace{g(0-y)}_{\overline{f(y)}}\, dy = \int_{-\infty}^{\infty} |f(y)|^2\, dy$$

Similarly, $\quad g(x) = \overline{f(-x)}$

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(x) \cdot e^{-ix\omega}\, dx = \int_{-\infty}^{\infty} \overline{f(-x)}\; e^{-ix\omega}\, dx$$

$-x = \theta$

$$= \int_{+\infty}^{-\infty} \overline{f(\theta)}\; e^{i\theta\omega}\; (-d\theta)$$

$$= \overline{\int_{-\infty}^{\infty} f(\theta)\, e^{-i\theta\omega}\, d\theta} = \overline{\hat{f}(\omega)}$$

$$f * g(x) = \int_{-\infty}^{\infty} \widehat{(f * g)}\,\omega \quad e^{ix\omega}\, d\omega$$

$$= \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot \hat{g}(\omega) \cdot e^{ix\omega}\, d\omega$$

$$f * g\,(\omega) = \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot \overline{\hat{f}(\omega)}\, dx = \int_{-\infty}^{\infty} f($$