2024/09/06 – Transform Techniques – Week 01

- vast applications

Fourier Series

Fourier Transform

Laplace transform

continuous

"f" sufficiently well-behaved f^n

$\longrightarrow$ f is $2L$ periodic, $f(x+2L) = f(x) \quad \forall x$

series of ∞ terms

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] = g(x) \quad (1)$$

Series of this form is called Fourier series

- very fundamental series

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx ,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

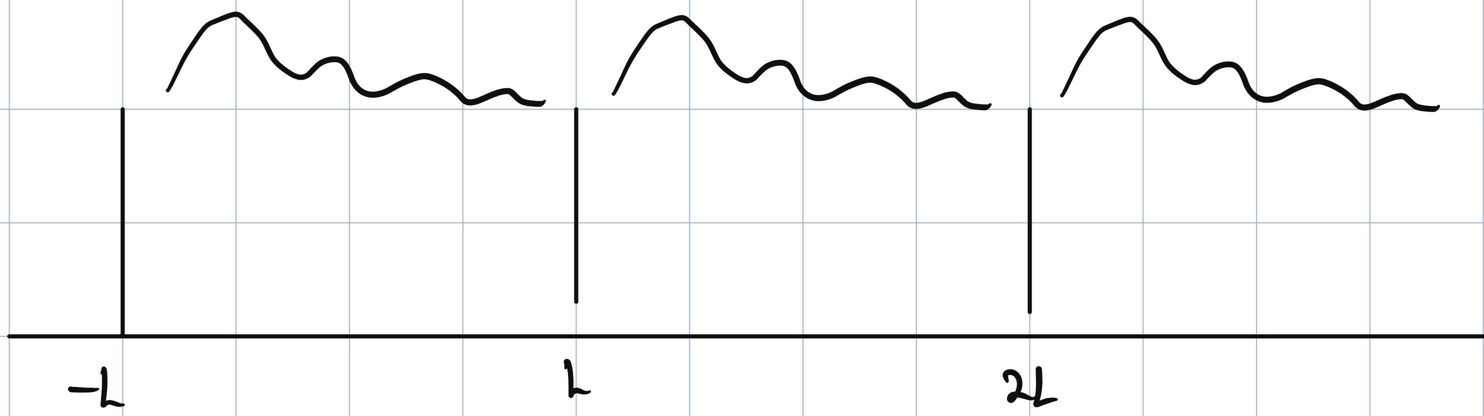
① $g(x) \rightsquigarrow$ when is this function meaningful

What is the guarantee that series converges?

② Sum of series $= f$?

if at all the series converges

What's the point of this?



for sufficiently big N

$x \in [-L, L]$

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

representation: $\{ a_0, a_n, b_n \}$

any transformation $\longrightarrow$ compress the data

whenever you need back your data $\longrightarrow$ decompress

Signals in real life are not periodic

Jugglesly

Orthogonality of the Trigonometric System

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{m\pi x}{L}\right)dx = \delta_{m,n}$$

Hint:

$\cos nx \cos mx = \frac{1}{2}\left[\cos(n+m)x + \cos(n-m)x\right]$

$\sin nx \sin mx = \frac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right]$

$\sin nx \cos nx = \frac{1}{2}\left[\sin(n+m)x + \sin(n-m)x\right]$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{m\pi x}{L}\right)dx = 0 \qquad \forall\, m, n$$

$$\left\{1,\ \sin\left(\frac{m\pi x}{L}\right),\ \cos\left(\frac{m\pi x}{L}\right)\right\}_{n=1}^{\infty}$$

basis

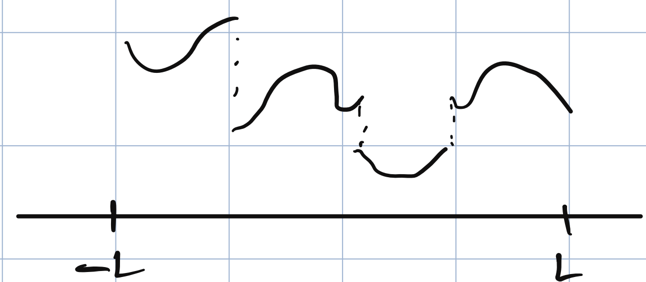
Dirichlet condition

1) f is $2L$ periodic. $f(x+2L) = f(x) \quad \forall x$

2) f is defined everywhere in $[-L, L)$ except possibly at a finite number of points

e.g.: $\tan x$

3) f and f' are piecewise smooth in $[-L, L)$
continuous



When f satisfies Dirichlet conditions

The fourier series in (1) converges to $f(x)$

$$\frac{f(x+) + f(x-)}{2}$$ where x is the point of discontinuity for f

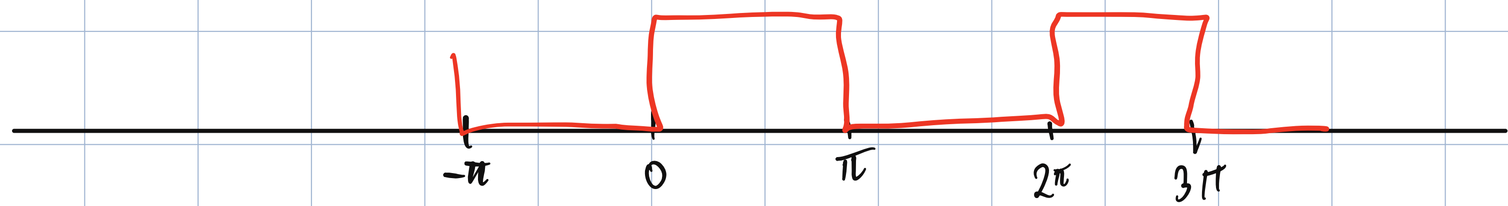
$$f(x+) = \lim_{\theta \to x^+} f(\theta) \text{ , (right limit)}$$

$$f(x-) = \lim_{\theta \to x^-} f(\theta) \text{ (left limit)}$$

Problem:

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi \le x < 0 \\ 1 & \text{if} \quad 0 \le x < \pi \end{cases}$$

and $f(x + 2\pi) = f(x) \quad \forall \; x \in (-\pi, \pi)$

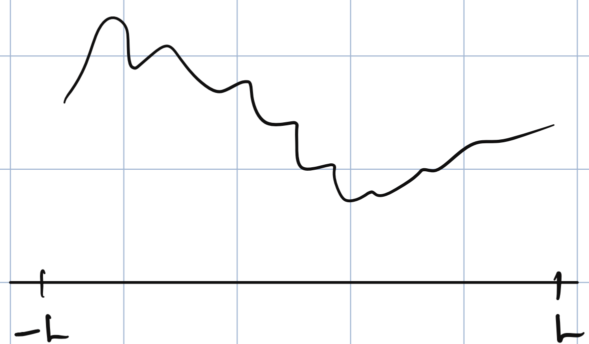


$a_n = 0 \quad \forall \; n$

$b_n = \dfrac{-1}{\pi n}$

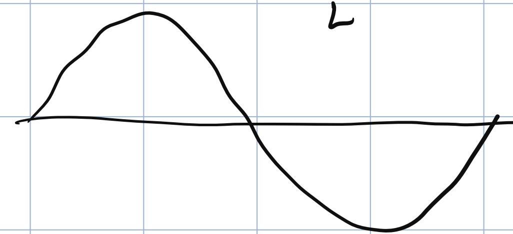
2024/09/09 – Transform Techniques

Fourier Series Expansion



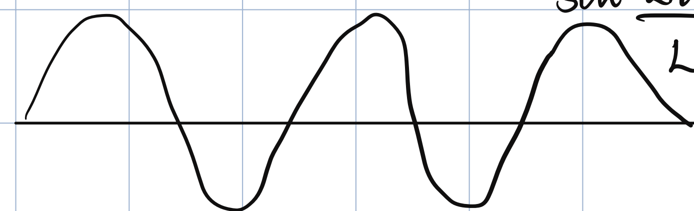
$=$

a_1 $\sin \frac{\pi x}{L}$



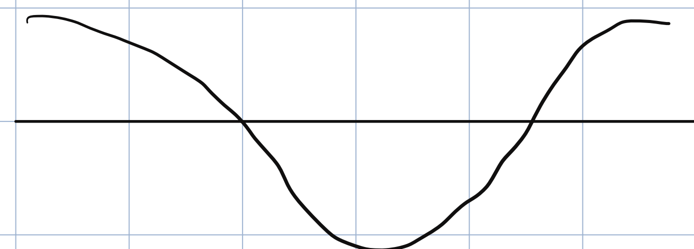
$+$

a_2 $\sin \frac{2\pi x}{L}$



$+$

b_0



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

usually large N to get decent app^x

$$f \sim \{a_0, a_n, b_n\}_{n=1}^{\infty}$$

$$g \sim \{\tilde{a}_0, \tilde{a}_n, \tilde{b}_n\}_{n=1}^{\infty}$$

$\sim$: f is convergent in the form of $\{a_0, a_n, b_n\}_{n=1}^{\infty}$

$g \to$ signal mimiced

$f, g \to$ which one is counterfeit?

* you cannot compare graphs directly

larger $L \Rightarrow$ more oscillations

$n \longrightarrow$ frequency

*

$$f \sim \underbrace{\{a_0, \ a_n, \ b_n\}}_{\text{compression} \quad \text{JPEG}}$$

What is the guarantee that the series is convergent?

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

What is the guarantee that $g(x) = f(x)$?

Dirichlet conditions $\Longrightarrow$ g is meaningful

$g(x) = f(x) \quad \forall\ x$

Textbook : Advanced Engg. Maths Erwin Kreyszig

If f is symmetric or even (i.e., $f(x) = f(-x)$)

$$b_n = \frac{1}{L} \underbrace{\int_{-L}^{L} \underbrace{f(x)}_{\text{even}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\text{odd}} dx$$

$\therefore\ b_n = 0 \quad \forall\ n$

$$\Rightarrow \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \longrightarrow \boxed{\text{fourier cosine series}}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

If f is odd or anti-symmetric i.e., $f(x) = -f(x)$

$$a_n = 0 \quad \forall\, n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{L} \int_0^1 f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

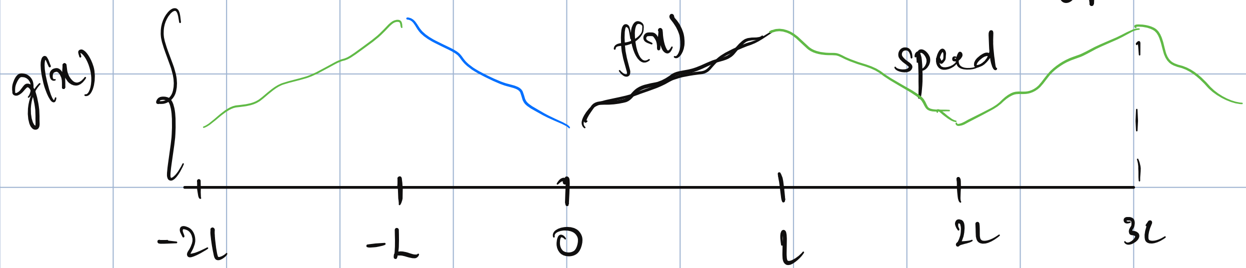
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) dx \longrightarrow \boxed{\text{fourier sine series}}$$

Half range expansion

$f \rightarrow -L$ to L } for mathematical purposes

real life

$\hookrightarrow f \rightarrow 0$ to L } $\rightarrow$ practically more meaningful



symmetric extension

$\downarrow$

$f(-x) = f(x)$

$x \in [-L, 0)$

$$g(x) = f(x) \quad \forall \quad x \in [0, L)$$

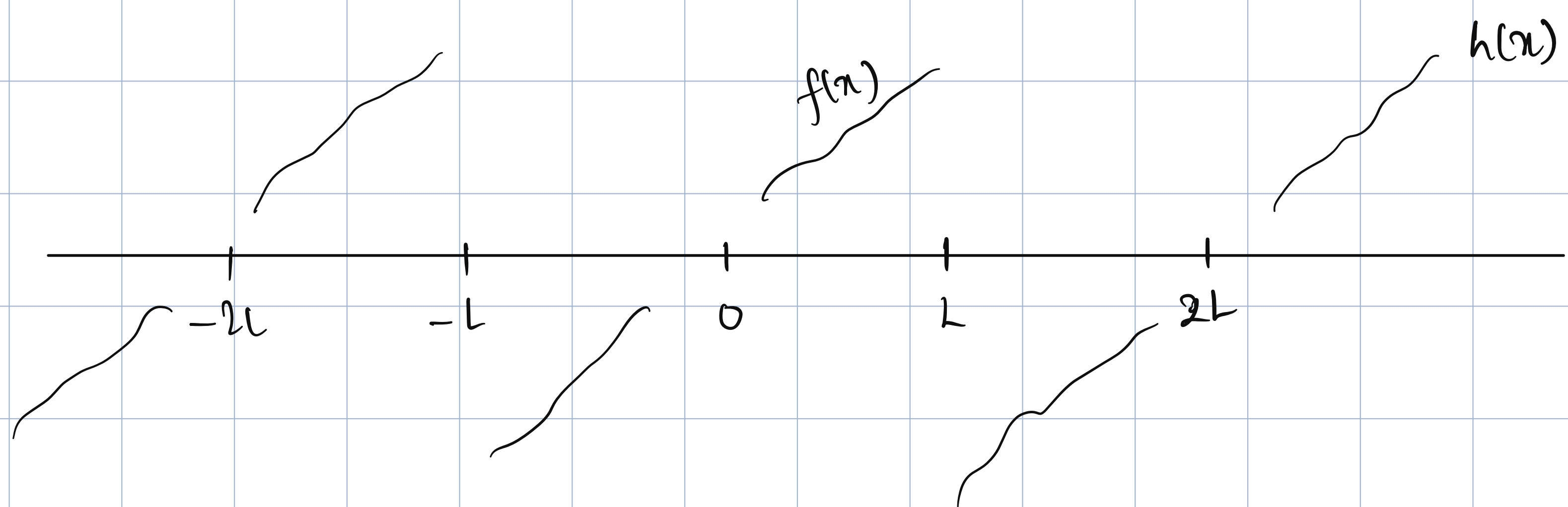
Because of symmetry

$$g(x) = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \forall \quad x \in (-\infty, \infty)$$

$$g(x + 2L) = g(x)$$

$$f(x) = g(x) = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos\left(\frac{n\pi x}{L}\right), \quad x \in [0, 1)$$

Half range expansion



$$h(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \forall \; x \in (-\infty, \infty)$$

$$h(x+2L) = h(x)$$

$$f(x) = \underbrace{h(x)\Big|_{[0,L)}}_{\text{restricted from 0 to L}} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \; \forall \; x \in [0, L)$$

$$h(x) = g(x) = f(x)$$

both expansion must result in same no.

satisfies Dirichlet condition

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$\boxed{\frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)}$$

fundamental result

energy remains same even if you go into another domain

$\int c \cdot c = m$

$\int c \cdot s$

$\int s \cdot c$

$\int c \cdot s$

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx = \frac{1}{L}\int_{-L}^{L}$$

Linear Algebra

$u \cdot v = 0$

discretis-ation vector

$\int f(x) \cdot g(x) = 0$

orthogonal

generalization

$$\frac{1}{2L}\int_{-L}^{L} e^{i\dots} \cdot e^{-i\dots} = \delta_{m,n}$$

$$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\cdot\cos\left(\frac{-m\pi x}{L}\right) = 0$$

$\forall\, m, n$

Application: solving PDE → Partial Differential Equation

$$\frac{\partial u}{\partial t} = 3\,\frac{\partial^2 u}{\partial t} , \quad t > 0 , \quad 0 < x < 2$$

$$u(0,t) = 0$$

$$u(2,t) = 0 \qquad \forall\, t > 0$$

$$u(x,0) = x \qquad \forall\, x \in (0,2)$$

$$u(x,t) = \underbrace{X(x)}_{f^n \text{ of } x} \; \underbrace{T(t)}_{f^n \text{ of } t}$$

From given : $\dfrac{\partial u}{\partial t} = X_x \cdot T_t'$

$$\frac{\partial^2 u}{\partial x^2} = X_x'' \cdot T_t$$

$$X_x \cdot T_t' = 3 X_x'' T$$

why const.

$$\underbrace{\frac{X_x}{X_x''}}_{f(x)} = \underbrace{\frac{T'}{3T_t}}_{g(t)} = k$$

Case ii) suppose $k > 0$, $k = p^2$ for some p

$$\therefore \frac{X''}{X} = p^2 = \frac{T'}{3T}$$

$$\therefore X'' - p^2 X = 0 \Rightarrow X(x) = c_1 e^{px} + c_2 e^{-px}$$

$$T' - 3p^2 T = 0 \Rightarrow T(t) = c_3 e^{3p^2 t}$$

$u(0,t) = 0$
$u(2,t) = 0$
$u(x,0) = x$

$$u(x, t) = X(x) \cdot T(t) = e^{3p^2 t}\left[a e^{px} + b e^{-px}\right]$$

$$u(0,t) = e^{3p^2 t}\left[a + b\right]$$

$$\neq 0$$

contradiction

Case iii) Suppose $k = -p^2$

2024/09/10

When f is symmetric / even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(t)\, e^{-inx}$$

When f is odd

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{L}\right)$$

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problem contd

$$\frac{X''}{X} = \frac{T'}{3T} = -p^2$$

$$T(t) = c_1 e^{-3p^2 t}$$

$$X''(x) = c_2 \cos(px) + c_3 \sin(px)$$

$$u(x,t) = c_1 e^{-3p^2t}\left[c_2\cos(px) + c_3\sin(px)\right]$$

$c_2 = a, \quad c_3 = b$

$$u(0,t) = 0 = e^{-3p^2t}\left[a + b(0)\right]$$

$$\Rightarrow \boxed{a = 0}$$

$$u(2,t) = 0 = e^{-3p^2t}\left(b\cdot\sin(p\cdot 2)\right) \Rightarrow 2p = n\pi$$

$$\Rightarrow p = \frac{n\pi}{2}$$

$$u(x,0) = b\sin\left(\frac{n\pi}{2}\right) = x$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n \, e^{-p^2 t} \sin\left(\frac{n\pi x}{2}\right)$$ linear combination

Consider u to be periodic

Since $u(x, 0) = x$

$$x = \sum_{n} b_n \sin\left(\frac{n\pi x}{2}\right)$$

Write later

Problems:

① Obtain the fourier series for $f(x) = e^{-x}$, $x \in [0, 2\pi]$

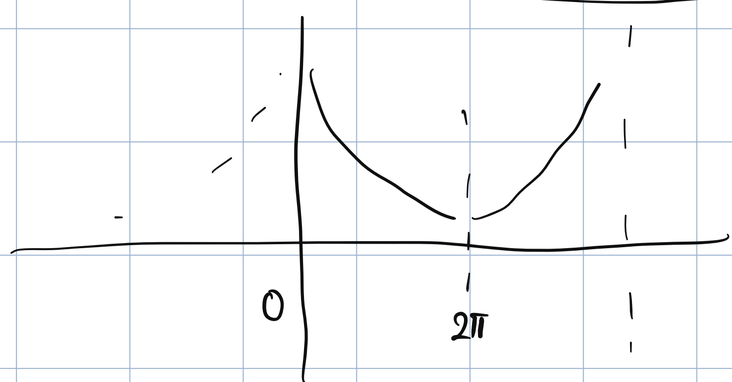
② Solve the equation, subject to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0$$

$$u(6,t) = 0$$

$$u(x,0) = \begin{cases} 1 & 0 \le x \le 3 \\ 0 & 3 < x < 6 \end{cases}$$



$$f(x) = f(-x)$$

$$a_n = \sum_{n=1}$$

2024/09/12

Recap:

$$f(x) = \sum_{n} c_n e^{inx}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-inx} f(x)\,dx$$

where f satisfies dirichlet condition (if not there is no guarantee about the convergence)

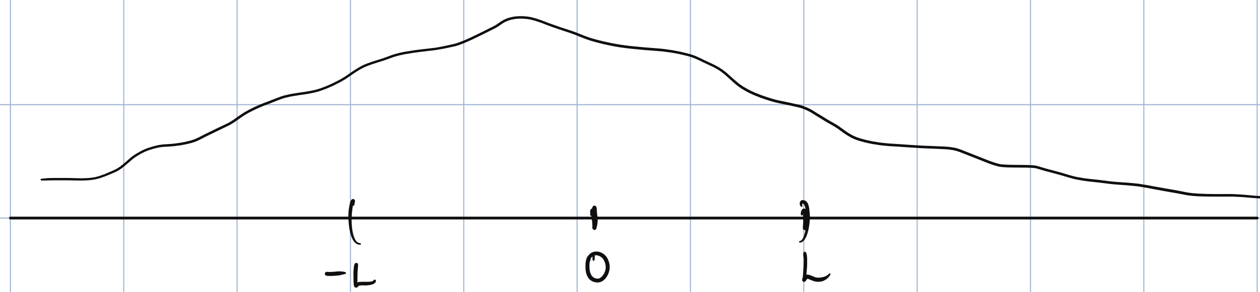
f is $2L$ periodic

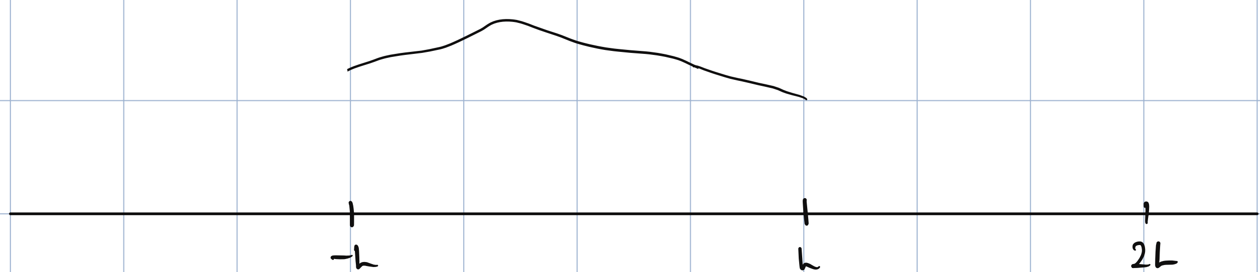
<u>What if the function is non-periodic?</u> → e.g. x^2 defined on all $\mathbb{R}$

periodic ← e^{ix}

periodic ← $\Big\{ \underbrace{e^{inx}}_{\text{basis}} \Big\}_{n=-\infty}^{\infty}$ → span ; linear combination

f is aperiodic, defined over $(-\infty, \infty)$



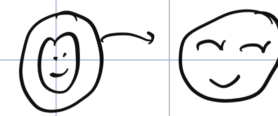


$$g(x) = f(x) \quad \forall \quad x \in [-L, L)$$

image processing, etc → non-harmonic expression

some real life applications
↓
gave rise to forming these results
(harmonic, non-harmonic expansion)

e.g.: Photoshop



↓ non-harmonic expansions
↓ more general basis

harmonic ~ sine, cosine
→ periodic basis (loose defⁿ)

$\underline{f}$ is known to you.
signal But you can identify hidden patterns thru coeff.

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \right]$$

$$= \frac{a_0}{2} + \sum_n \Bigg\{ \overbrace{\frac{1}{L} \int_{-L}^{L} \left(f(t) \cos\left(\frac{n\pi t}{L}\right) dt \right) \cos\left(\frac{n\pi x}{L}\right)}^{\text{cosine term}} + \underbrace{\left(\frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt \right) \sin\left(\frac{n\pi x}{L}\right)}_{\text{sine term}} \Bigg\}$$

$$\frac{n\pi}{L} = \alpha_n$$

cosine term

$$= \sum_n \underbrace{\frac{1}{\pi} \left(\int_{-L}^{L} f(t) \cos(\alpha_n t)\, dt \right) \cos(\alpha_n x)}_{h(\alpha_n)} \cdot \Delta \alpha_n$$

$$\Delta \alpha_n = \frac{\pi}{L}$$

$$= \sum_n h(\alpha_n) \cdot \Delta\alpha_n$$

$$\longrightarrow \int_0^\infty h(\alpha)\, d\alpha \quad = \quad \int_0^\infty \left[\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\alpha t)\, dt \right] \cdot \cos \alpha x \; d\alpha$$

$$|a_0| = \left| \frac{1}{L} \int_{-L}^{L} f(x)\, dx \right| \leq \frac{1}{L} \int_{-L}^{L} |f(x)|\, dx$$

$$\leq \frac{1}{L} \int_{-\infty}^{\infty} |f(x)|\, dx$$

Suppose f is such that

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

is finite

then $|a_0| \to 0$ as $L \to \infty$

$$f(x) = \int_0^\infty \left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha$$

$g(x) = f(x)$ from $x = -L$ to L ; as $L \to \infty$, $g(x) = f(x)$

Suppose
$$\left.\begin{cases} A(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(t)\cos(\alpha t)\,dt \\[2ex] B(\alpha) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(t)\cdot\sin(\alpha t)\,dt \end{cases}\right\}$$

Fourier integrals

Forward transform

freq. domain $\leftarrow \int$ time domain function

Inversion $\rightarrow$
$$f(x) = \int_0^\infty \left(A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right)d\alpha$$

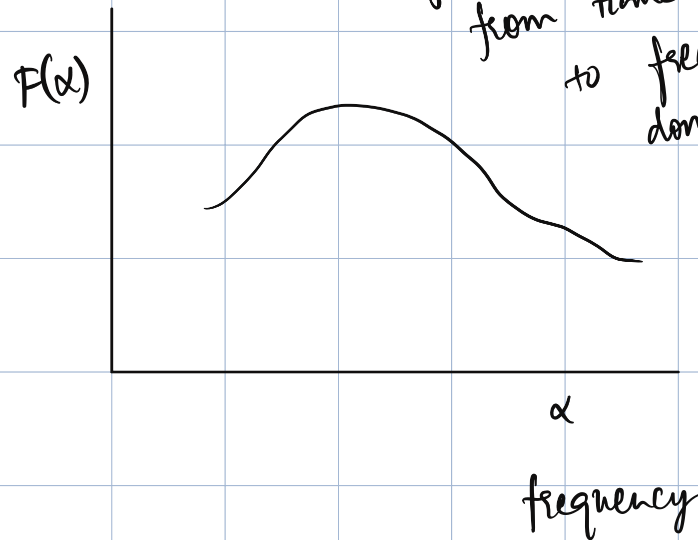
$$f(x) = \int_0^\infty \left[\left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \alpha t \ dt \right) \cos \alpha x + \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \alpha t \ dt \right) \sin \alpha x \right] d\alpha$$

$$= \frac{1}{\pi} \underset{(\alpha)}{\int_0^\infty} \underset{(t)}{\int_{-\infty}^{\infty}} f(t) \cdot \underbrace{\left[\cos \alpha t \cos \alpha x + \sin \alpha t \sin \alpha x \right]}_{\cos[\alpha(x-t)]} dt \quad d\alpha$$

$$= \frac{1}{2\pi} \underset{(\alpha)}{\int_{-\infty}^{\infty}} \underset{(t)}{\int_{-\infty}^{\infty}} f(t) \left[\cos \alpha (\right.$$

transform

hidden, non-trivial patterns can be identified by moving from time domain to frequency domain



Stock market, etc.

Problems from book

$L = 2\pi$

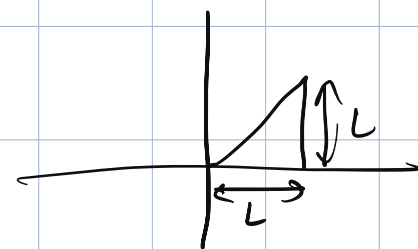
$\frac{n\pi x}{L}$

$nx \rightarrow \frac{1}{\pi}$

$f(x) = |x|$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{a_0}{2} = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$



$$\frac{a_0}{2} = \frac{1}{2L} \times L^2 = \frac{L}{2}$$

$$\boxed{a_0 = L = 2\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{|x|}_{\text{even}} \underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}\, dx$$

$$= \frac{1}{L} \times 2 \int_0^L x \cdot \cos\frac{n\pi x}{L}\, dx$$

$$a_n = \frac{2}{n\pi}\left(\cos n\pi - 1\right)$$

$n = \text{even} \Longrightarrow$

$$\boxed{\begin{aligned} a_n &= 0 \\ \underset{\text{odd}}{a_n} &= \frac{-4}{n\pi} \end{aligned}}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} |x| \underbrace{\sin\left(\frac{n\pi x}{L}\right)} dx = 0$$

$$a_n = \frac{2}{L}\int^{L} x\cos\frac{n\pi x}{L}\, dx$$

ILATE

$$\int uv = u\int v - \int u'\int v$$

$$= \cancel{x\,\frac{\sin(n\pi x/2)}{n\pi/2}} - \int_0^L \sin\frac{n\pi x}{L}$$

$$= \frac{\cos\left(\frac{n\pi x}{L}\right)}{\left(n\pi/2\right)}\Bigg]_0^L$$

$$= \frac{L}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= \frac{2}{L}\left\{\left[\frac{Lx}{n\pi}\left(+\sin\frac{n\pi x}{L}\right)\right]_0^L - \int \sin\frac{n\pi x}{2}\,dx \cdot \frac{L}{n\pi}\right\}$$

$$= \cancel{\frac{2}{L}\left\{\frac{L\cdot L}{n\pi}\sin(n\pi) - 0\right\}} \to 0$$

$$+ \frac{2}{\cancel{L}}\left\{\frac{L^{\cancel{2}}}{n^2\pi^2}\left(\cos\left(\frac{n\pi \cancel{L}}{\cancel{L}}\right) - 1\right)\right\}$$

$$= \frac{2\cdot 2\cancel{\pi}}{n^2\pi^{\cancel{2}}}\left(\cos(n\pi) - 1\right)$$

$n \in$ even ; $a_n = 0$

$n \in$ odd ; $a_n = -8$

$n = \text{odd}$; $c_n = \frac{8}{\pi n^2}$

$$\left(\text{If } \frac{1}{2L} : \frac{-4}{n^2 \pi} \right)$$

$$c_n = \frac{1}{T} \int_0^T f(t)\, e^{-\frac{2\pi i n t}{T}}\, dt$$