$$X = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow \text{no remainder}$$

$$X' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \rightarrow \text{produce remainder}$$

$$Y = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \rightarrow \text{propagate remainder}$$

$$Y' = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow \text{consume remainder}$$

$B \longrightarrow$ some Xs

then

exactly one from X'

then some Ys

then exactly one Y'

1 1

1 1

0 0

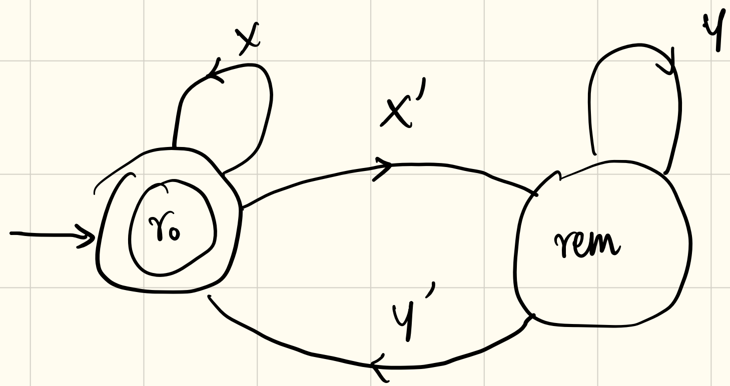
0 1 0

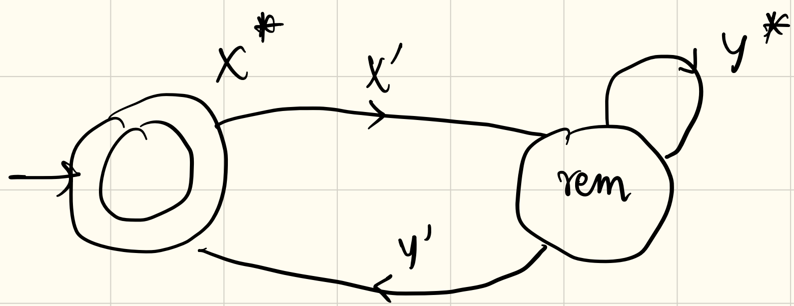
0

1 1 1

0

1





$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{matrix} 1 \\ 1 \end{matrix}$$

$$\begin{matrix} 0 \\ 0 \end{matrix} \longrightarrow \text{remainder } 0$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{matrix} 0 * 3 \\ 1 \end{matrix} \quad \text{remainder } 1 \qquad \begin{matrix} 1 & 0 \\ & 1 \end{matrix}$$

$$\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 3 \\ 1 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$$

1 0 0 1

1 0 1 0

0

1

1

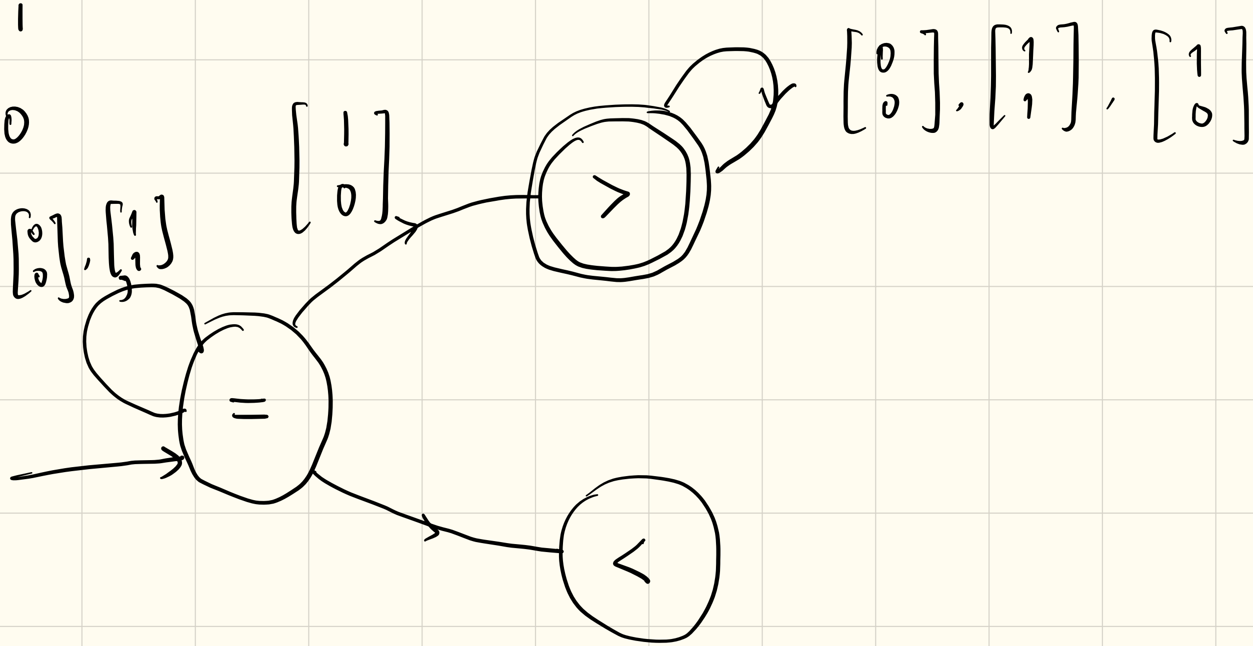
0

1 0

1 0

0 1 0

1 0 0



Assume E is regular.

Take $p \in \mathbb{N}$ to be its pumping length

let

$$w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^p \begin{bmatrix} 1 \\ 0 \end{bmatrix}^p \in E$$

xyz

s.t. $|xy| \leq p$

$\therefore$ y can only contain $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence, xy^2z clearly does not belong to E.

This is a contraction the language being regular.

$B_n = \{ a^k \mid k \text{ is a multiple of } n \}$

$B_1 = \{ a^k \mid k \text{ is a multiple of } 1 \}$

$= a\, a^*$

$B_2 = \{ a^k \mid k \text{ is even} \}$

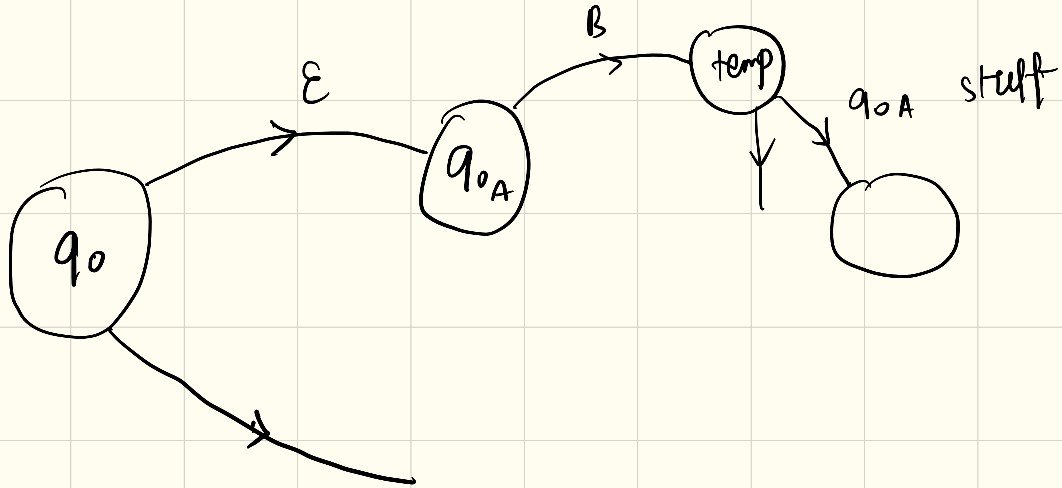
$= (aa)^*$

$B_3 = \{ a^k \mid k \text{ is multiple of } 3 \}$

$= (aaa)^*$

1.37. C_n

$a_1 \quad b_1 \quad a_2 \quad b_2 \quad a_3 \quad b_3 \quad \ldots \quad a_k \quad b_k$



k

Let $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$

$M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$

$$M_s = (Q_A \cup \{t_1, \dots t_k\}, \delta', q_0, F_A')$$

$$\delta'(q, \alpha) = \begin{cases} t_i & \text{, if } q = q_i \\ \delta(q_i, \alpha) & \text{, if } q = t_i \end{cases}$$

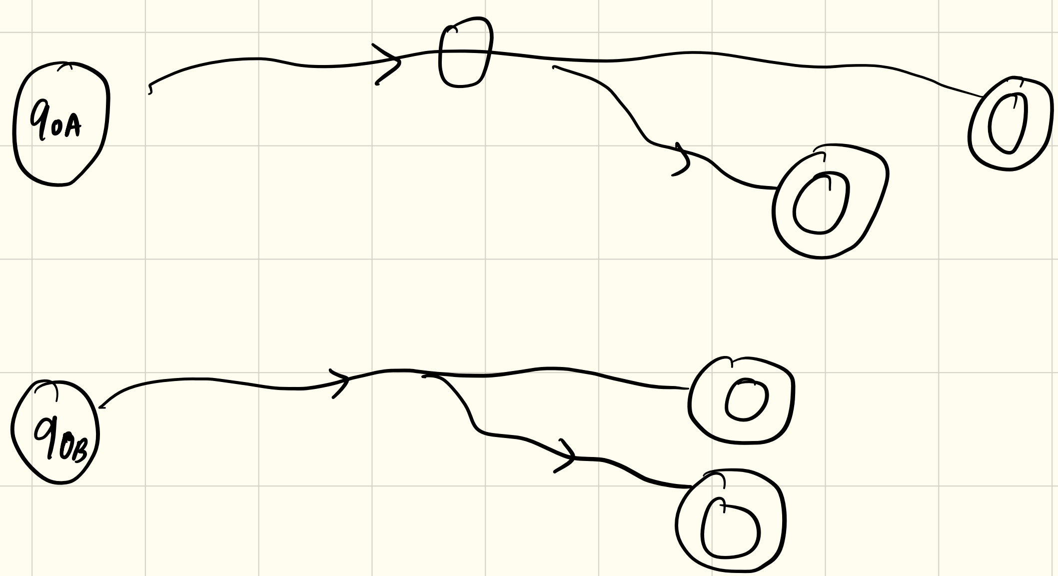
Let $a = a_1 \dots a_k$ be any string from A

Then by defn of acceptance

$$\hat{\delta}_{M_1}(q_0, a) \in F$$

$a' = a_1 c_1 \dots a_k c_k \quad , \quad c_i \in \Sigma$

$$\hat{\delta}_{M'}(q_0', a') \in F$$



jump between machines M_1

$$M = (Q_1 \times Q_2,\ \Sigma,\ \delta,\ (q_1, q_2),\ F_1 \times F_2)$$

$\boxed{\text{NFA}}$

$$\delta((q_1, q_2), \alpha) = \{(\delta_1(q_1, a), q_2),\ (q_1, \delta_2(q_2, a))\}$$

Proof: Let $a_1 a_2 \dots a_k$ be any string from A,

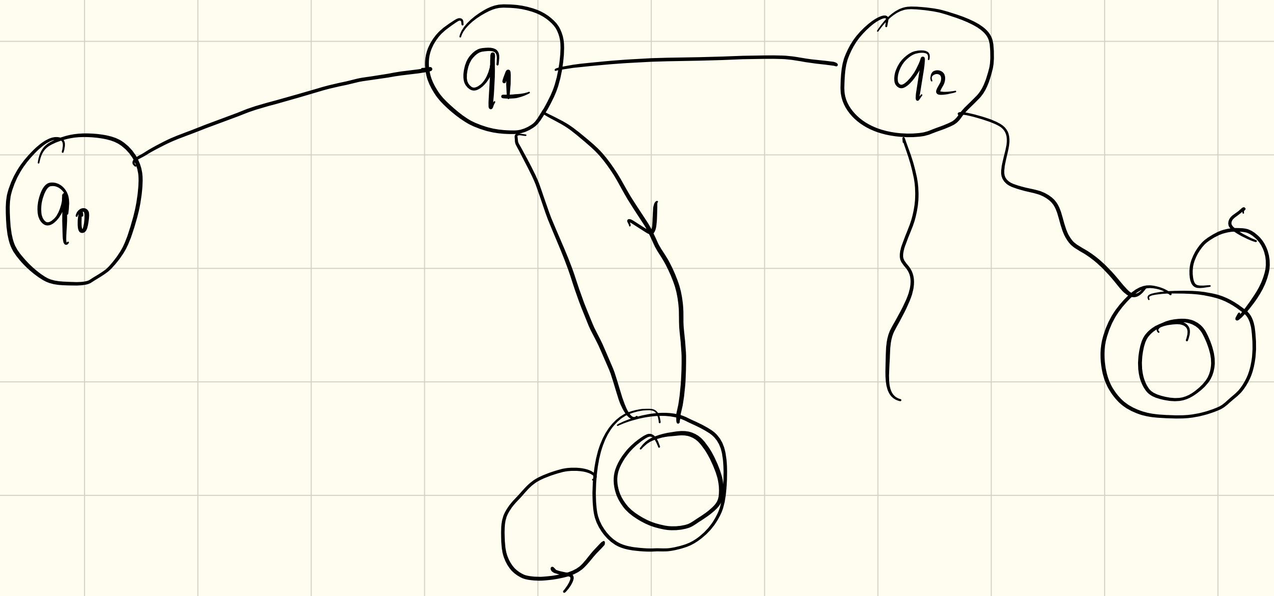
then by def^n of M_1 $\qquad a_i \in \Sigma^*$

$\exists$ sequence of states

$r_0, r_1, \dots, r_k \in Q_1$

such that $r_0 = q_1$, $r_k \in F_1$

and for each $i = 0 \dots k-1$: $\delta_1(r_i, a$



Copy the whole machine which accepts A , add arrows from old machine to copied one.

$$M' = (Q \cup Q', \Sigma, \delta', q_0, F')$$

$$\delta'(t, \alpha) = \begin{cases} \left\{ \bigcup_{\beta \in \Sigma} \left(\delta(\delta(q, \beta), \alpha) \right) \right\} \cup \{ \delta(q, \alpha) \} & t = q \\ \{ (\delta(q, \alpha))' \} , & \text{if } t = q' \end{cases}$$

$w \in B$ s.t. for some $y \in C$

strings w and y contain equal no. of 1's

B and C parallel

$B \to$ usual

$C \xrightarrow{\text{NFA}}$ on input $1 \longrightarrow$ same as C

on input $0 \longrightarrow$ go to all states that can be reached with conseentive 0s.

$$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \rightarrow B$$

$$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \rightarrow C$$

$$M = (Q_1 \times Q_2 \ \Sigma, \delta', (q_1, q_2), F_1 \times F_2)$$

$$\delta((r, s), \alpha) = \begin{cases} U & \alpha = 0 \\ (\delta_1(r, \alpha), \delta_2(r_2, \alpha)) & \alpha = 1 \end{cases}$$

$w \mid wx \in A$ and $x \in B$

$\underline{0^p}\ 1\ 0^p$

$0^m 1^n$ suppose regular

$\Longrightarrow$ $0^m 1^n \cap 0$

$w \mid w \in \{0,1\}^*$ is a palindrome

$\underline{0^p}\ 1^p\ 1^p\ 0^p$

$Y = \{ w \mid w = x_1 \# x_2 \# \dots \# x_k$

$1^p \# 1^{p-1} \# 1^{p-2} \# \cdots \# 1^2 \# 1$

y must be 1^k

1 0 1 0 0 1

1 0 1 1

0 1

$1^k y \mid y \in \{0,1\}^*$ y contains atleast k 1s

$1 \quad \underline{0 \quad 1 \quad 1 \quad 1}$

0^p

$1 \quad \underline{\underline{\Sigma^* \quad 1 \quad \Sigma^*}}$

$x \equiv_L x$

$x \equiv_L y \implies y \equiv_L x$

$\Rightarrow \quad \forall z \in \Sigma^*$

$xz \in L \iff yz \in L$

$x \equiv_L y \qquad y \equiv_L z$

$a \; b^n \; c^n$

a^i

0 0 1

$0^* 1^+ 0^+ 1^* \cup 10^* 1$

1 1 cannot be pumped

1 0 1 0 1 0

01 1 0 0

1 0 1 0 0

1^n

$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32,$

$\downarrow \quad \downarrow \quad \quad \downarrow \quad \downarrow$

$1, \quad 2, \quad 11, \quad 21, \quad 121, \quad 1012$

$1 \times 3^2 + 3 \times 2 + 1$

$1^n \quad 2^n$

$1 \times 3^3 + 3 \times 1 + 2$

$2^n \quad 1^n$

$1 \quad 0 \quad 12$

2 1

7

1ⁿ

$1^n \quad 2^n$

1 2

5

2 2 1 1

1 + 3 + 18 + 54

$$\begin{array}{r} 22 \\ 54 \\ \hline 76 \end{array}$$

1 1 2 2

$$a = 2^n - 1$$

$$b = 2^m - 1$$

$$\underbrace{b - a}_{\text{divisible by } 2^n} = 2^m - 2^n$$

$$= 2^n \left(2^{m-n} - 1 \right)$$

$$xyyz - xyz = 3^{2|y|+|z|} \cdot x + 3^{|z|} \left(3^{|y|} \cdot + 1 \right) y$$

$$+ \cancel{z}$$

$$- 3^{|y|+|z|} x - \cancel{3^{|z|} y} - \cancel{z}$$

$$= \underbrace{3^{|y|+|z|}}_{\text{not dive}} \underbrace{\left(\left(3^{|y|} - 1\right) x + y \right)}_{\text{should be divisible by } 2^n = a+1}$$

But $a + 1 = xyz + 1 > xy$

$> xy - x$

A regular

$$B = \{ x \mid \exists y, \ |y| = |x|^2 \text{ and } xy \in A \}$$

$$N = (Q, \Sigma, \delta, q, F)$$

$$N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$$

$$Q_1 = Q_1 \times \mathcal{P}(Q)$$

$$= (q, S \mid q \in Q, S \in \mathcal{P}(Q))$$

$$q_1 = (q, F)$$

$$\delta_1((q, A), \alpha)$$

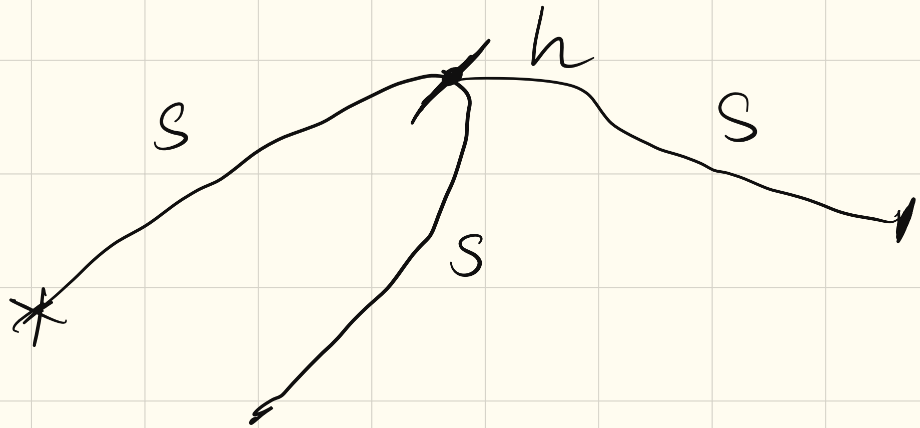
$$= \left(\delta(q, \alpha), \{ p \in Q \mid \exists a \in \Sigma, b \in S, \delta(p, a) = b \} \right)$$

$$F_1 = (q, S), \quad q \in S$$

$\{ 1^n x 1^n \mid n \in \mathbb{N} \} \longrightarrow$ not regular

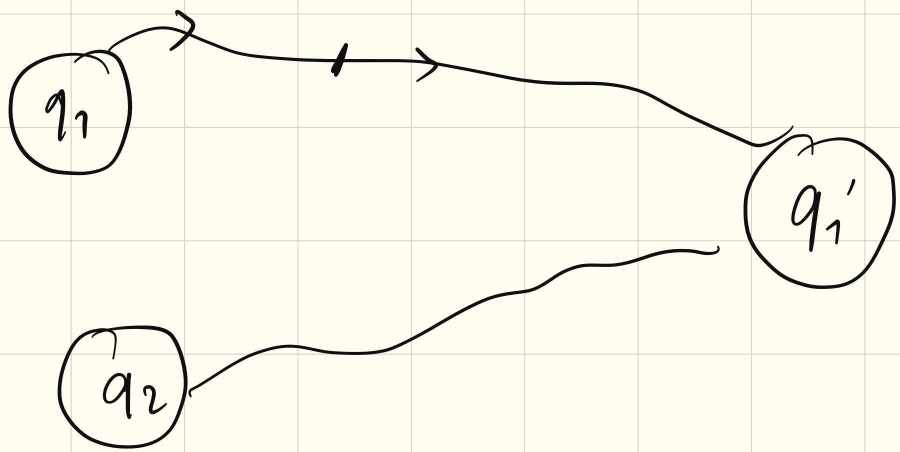
$\hookrightarrow$ contains 0

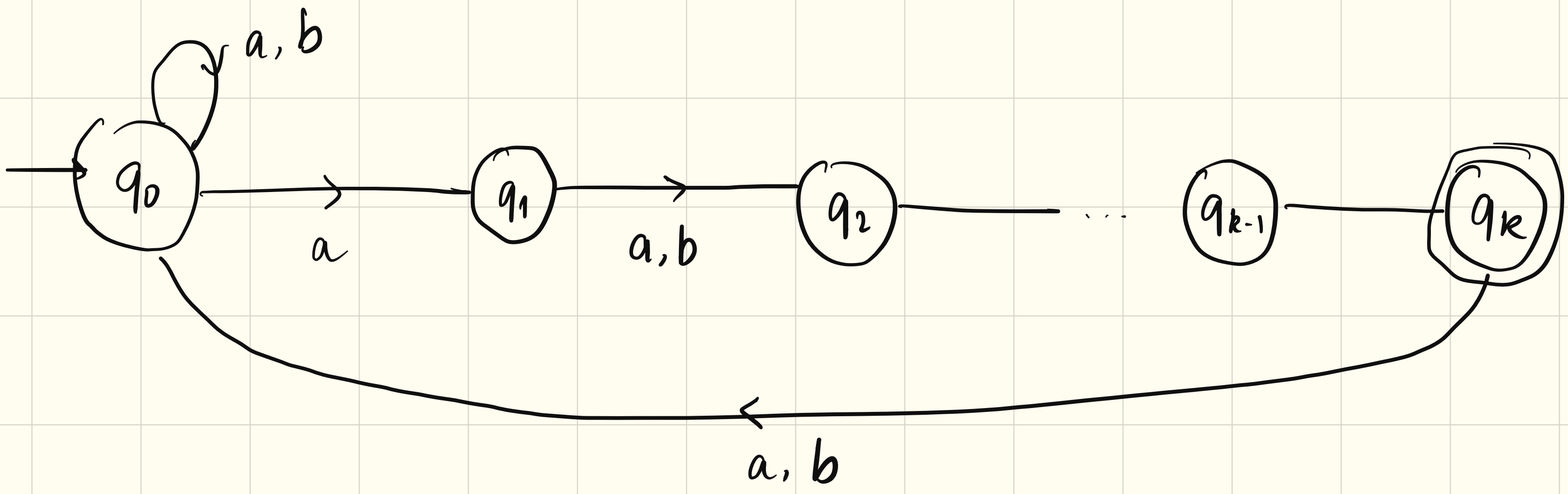
$(1^* 0)^* 1^*$


h
s
S
s

Arbitrary q_1, q_2 in a k-state synchronizable

$\exists$ a string s s.t. $\hat{\delta}(q_1, s) = \hat{\delta}(q_2, s)$.





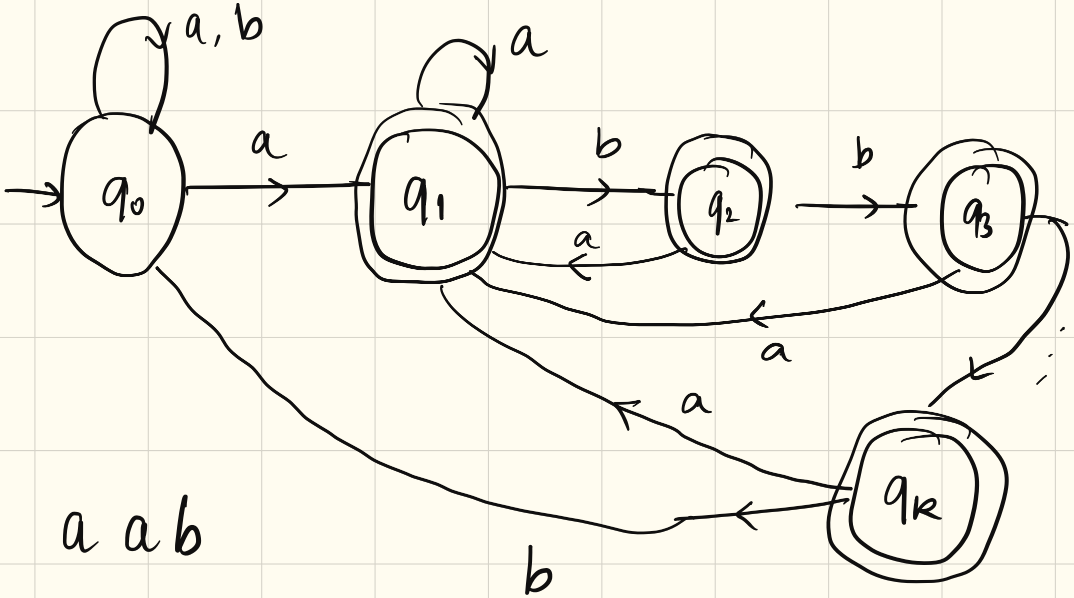
1^+ $\{1, 11, 111, \ldots\}$

Prove by Myhill - Nerode Theorem that index of C_k is 2^k $\Rightarrow$ Minimal no. of states required to recognize $C_k = 2^k$

Consider $X = \Sigma^k$, $\Sigma = \{a, b\}$ $|X| = 2^k$

<u>Claim</u>: X is pairwise distinguishable by C_k ($\Rightarrow$ index $\geq 2^k$)

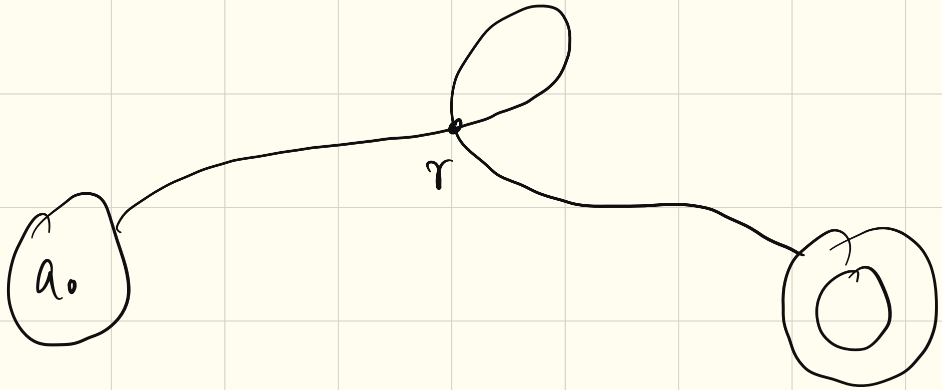
Take any two distinct $x, y \in X$, then they will differ at some position say at m. One will have



a a b

$$\delta(r, \alpha) = \begin{cases} q_0 & r = q_0 ,\ \alpha = a \text{ or } b \\ q_1 & r = q_0 ,\ \alpha = a \\ q_{n+1} & r = q_n \quad \alpha = b \\ & \qquad n = 1 \ldots k-1 \end{cases}$$

Disc → Blue
Reddit → Orange
Youtube → Red
Google
ChatGPT → cyan
Perplexity → green
Wikipedia
GitHub → purple
Gmail
Classroom



$A_e = \{ s \mid s \in A$, passes through r even no. of times$\}$

$A_o = \{ s \mid s \in A$, passes through r odd no. of times$\}$

$D_{even} = (Q_e, \Sigma, \delta_e, q, F_e)$

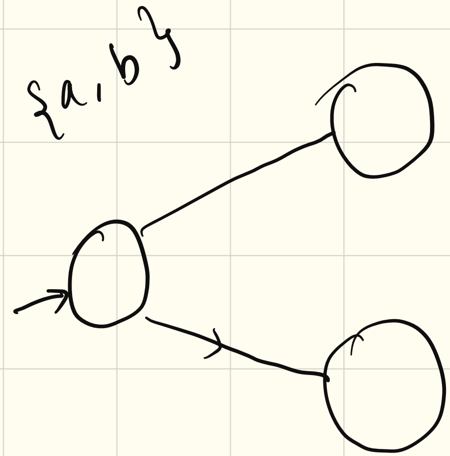
$Q_e = \{ p, q_{0e}, q_{0o}, q_{1e}, q_{1o}, \dots, q_{ke}, q_{ko} \}$

A has two disjoint regular subsets

$B \subseteq D$

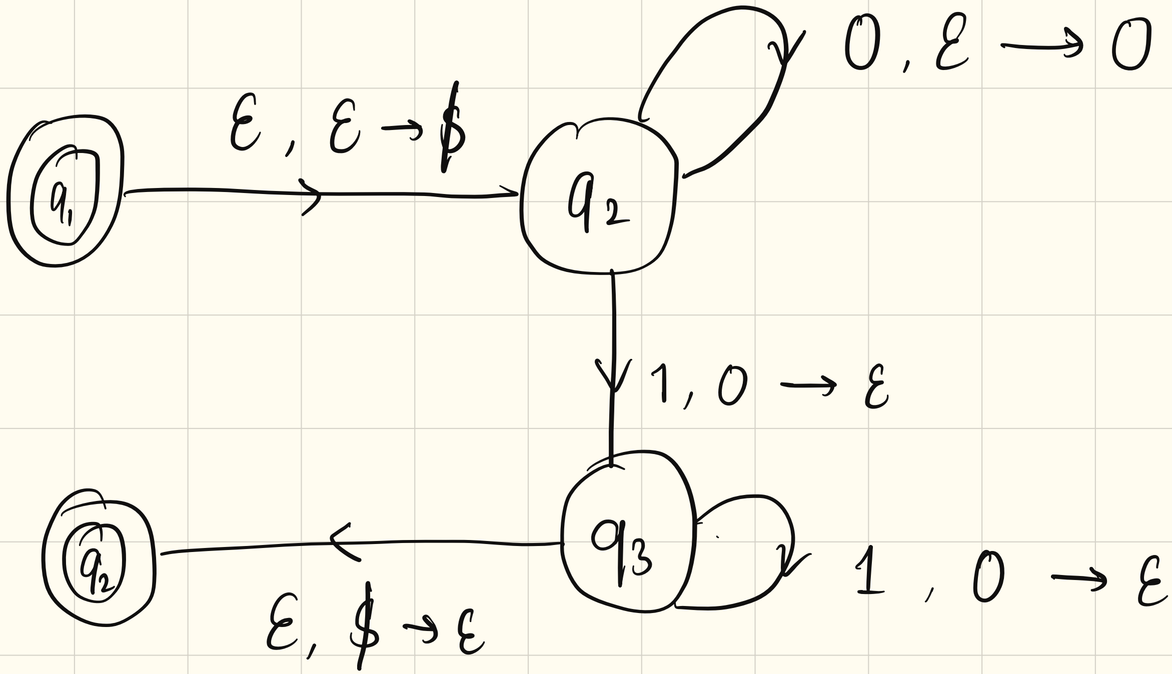
$D \setminus B$ is ∞

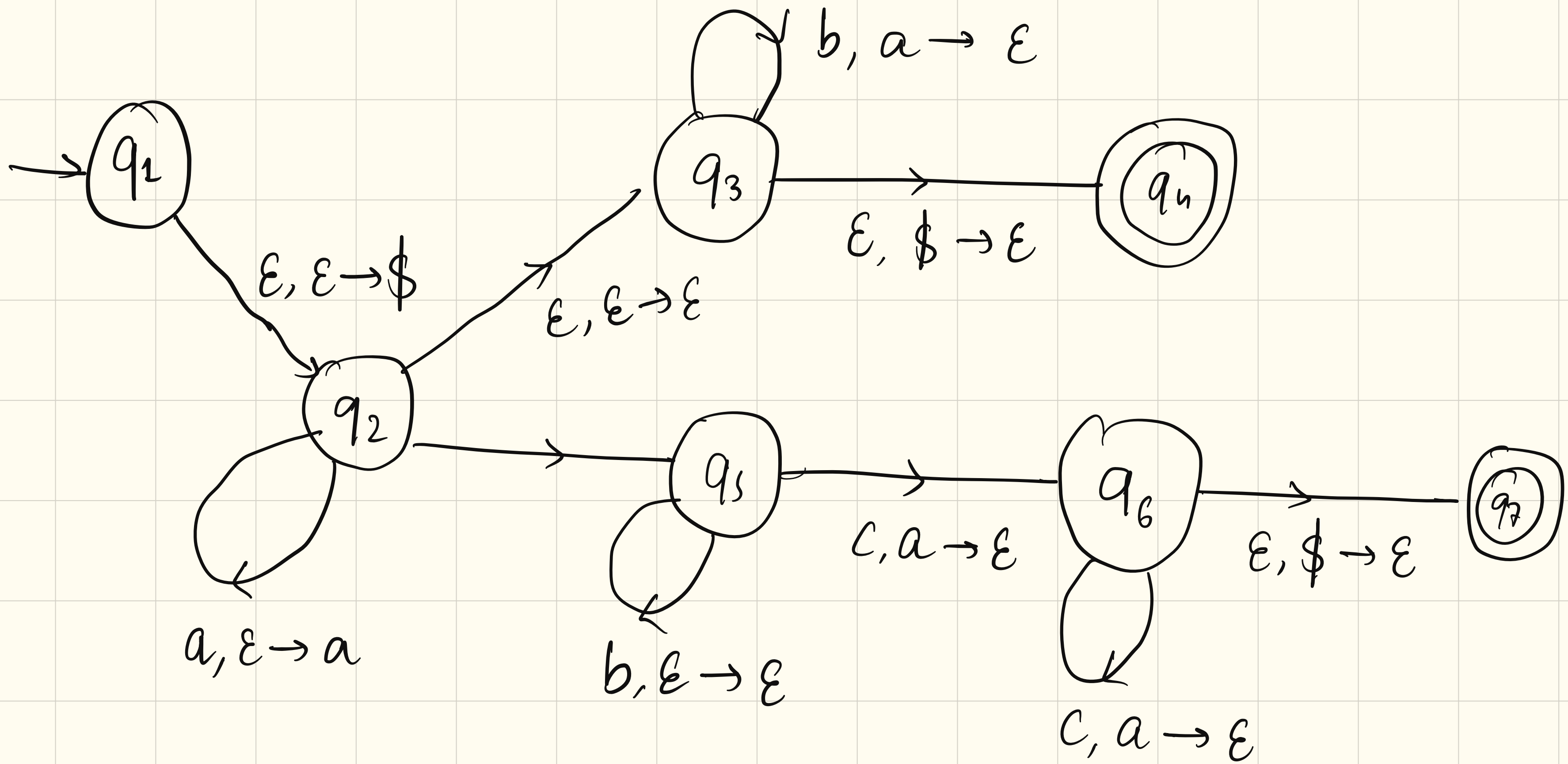
C $\qquad$ D



Ak

$0^n 1^n$




q_1
q_2
q_3
q_4
q_5
q_6
q_7
$\varepsilon, \varepsilon \to \$$
$a, \varepsilon \to a$
$\varepsilon, \varepsilon \to \varepsilon$
$b, a \to \varepsilon$
$\varepsilon, \$ \to \varepsilon$
$b, \varepsilon \to \varepsilon$
$c, a \to \varepsilon$
$c, a \to \varepsilon$
$\varepsilon, \$ \to \varepsilon$

$$s = u\,v\,w\,x\,y$$

$$S \rightarrow TS \mid 0 \mid 1$$

$$T \rightarrow 00 \mid 01 \mid 10 \mid 11$$

$$S \rightarrow X0Y \mid 0$$

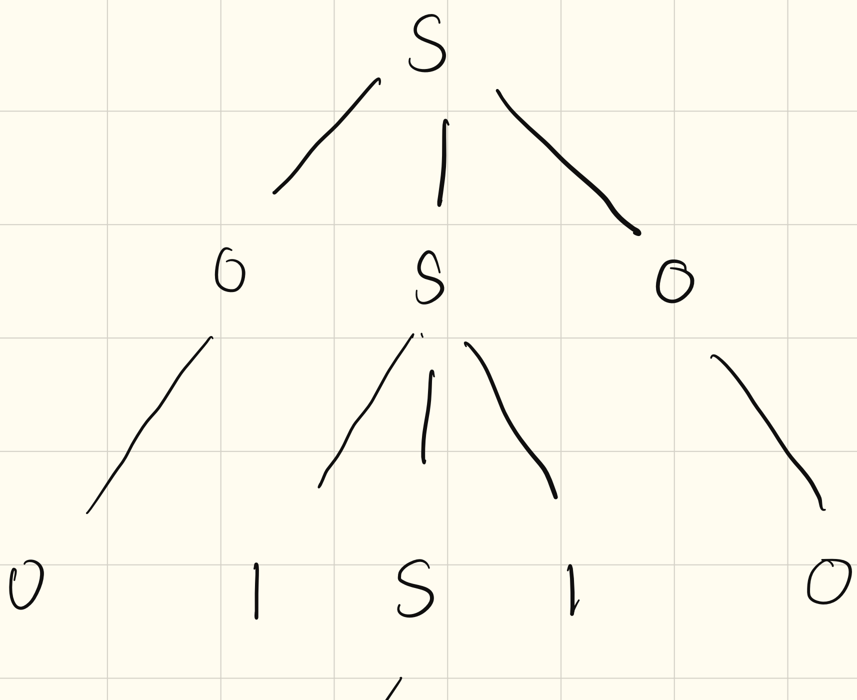
$x + y = \text{even}$

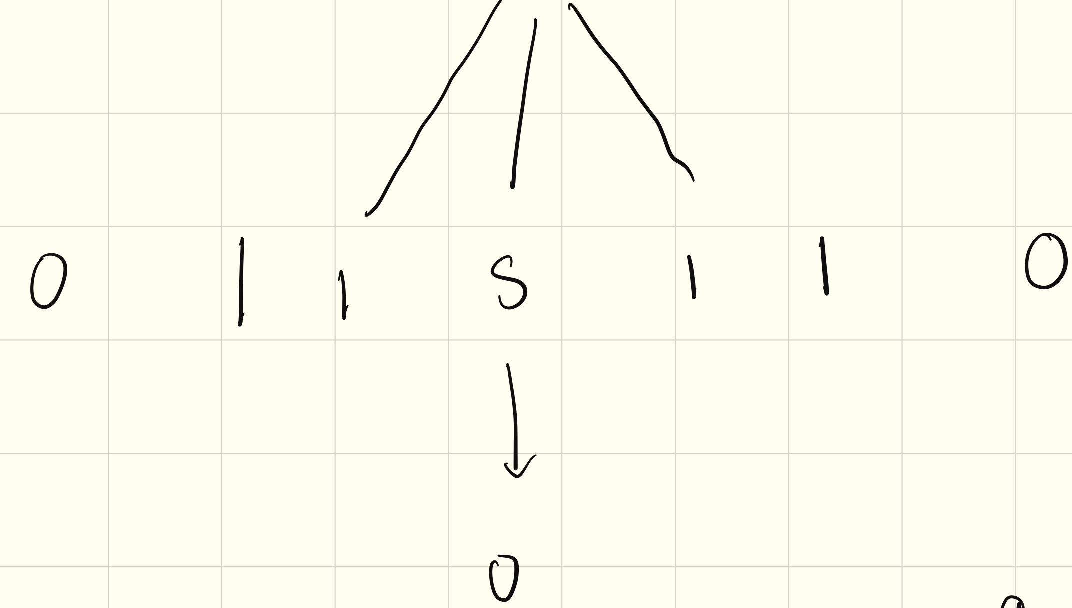
$$X \rightarrow$$

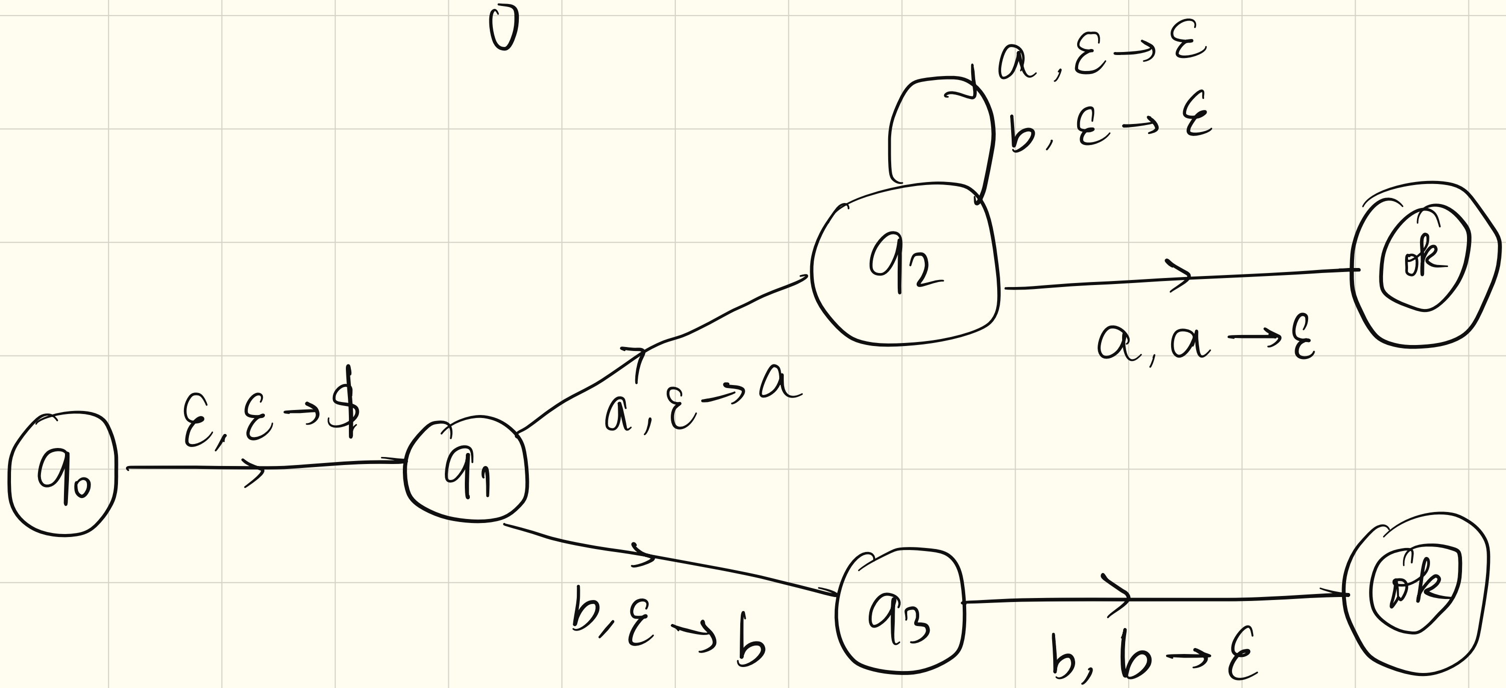
$$S \to 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid 0$$

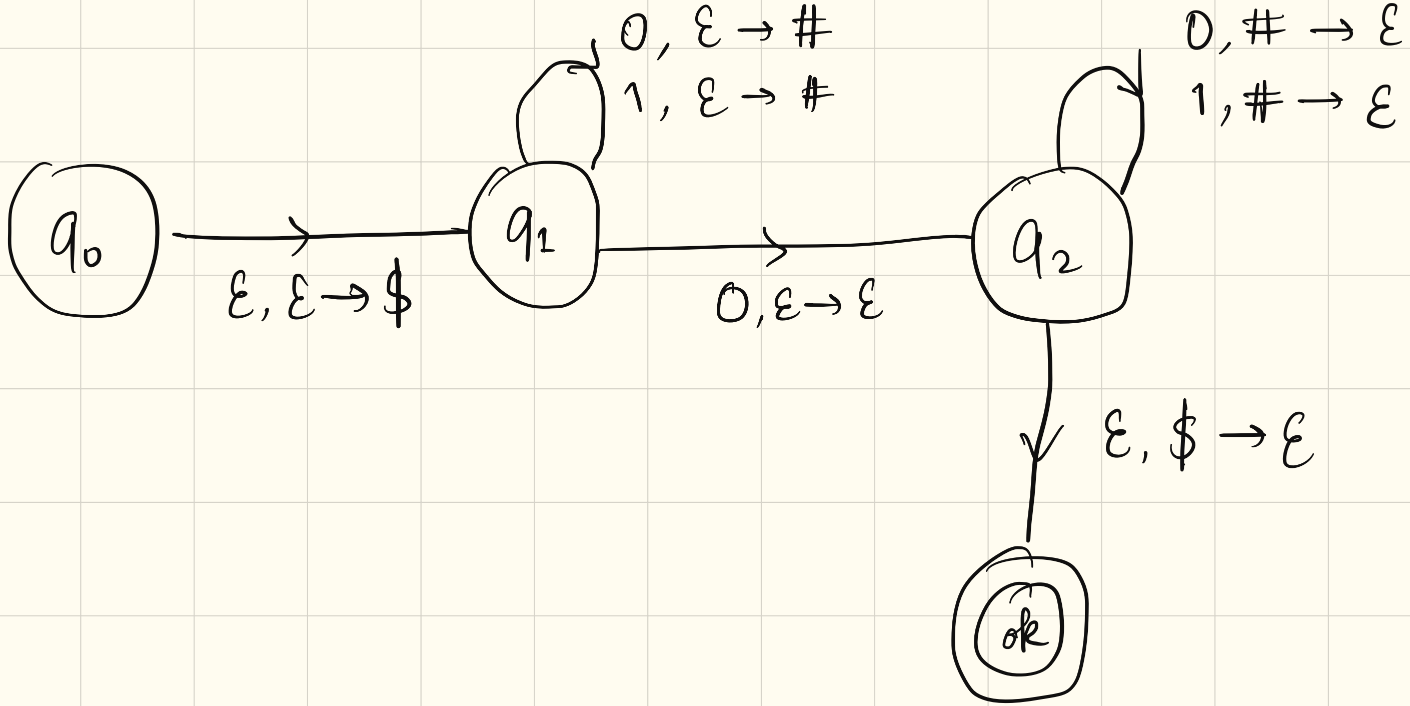
1

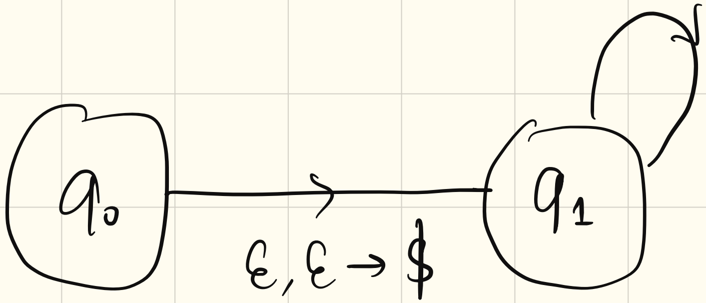
$$S \to 0S0 \mid 1S1 \mid 0 \mid 1$$




0 | 1 S 1 | 0
0


q0
ε, ε → $
q1
a, ε → a
b, ε → b
q2
a, ε → ε
b, ε → ε
q3
a, a → ε
b, b → ε
ok
ok


q_0
$\varepsilon, \varepsilon \to \$$
q_1
$0, \varepsilon \to \#$
$1, \varepsilon \to \#$
$0, \varepsilon \to \varepsilon$
q_2
$0, \# \to \varepsilon$
$1, \# \to \varepsilon$
$\varepsilon, \$ \to \varepsilon$
ok



$a^n \quad b^n$

$$S \to Sab \mid Sba \mid bSa \mid aSb \mid$$
$$baS \mid abS \mid aS \mid$$
$$Sa \mid a$$

$$S \to S_1 \mid S_2 \mid S_3$$

$$S_1 \to S_1ab \mid S_1ba \mid aS_1b \mid bS_1a \mid abS_1 \mid baS_1 \mid bS \mid Sb \mid b$$

S_2 ✓

$$S_3 \to Tab \mid bTa \mid baT$$

$$T \to Tab \mid Tba \mid aTb \mid bTa \mid abT \mid baT \mid \varepsilon$$

$R \to SX$

$w \# (0 \cup 1)^* w^R$

$S \to TX$

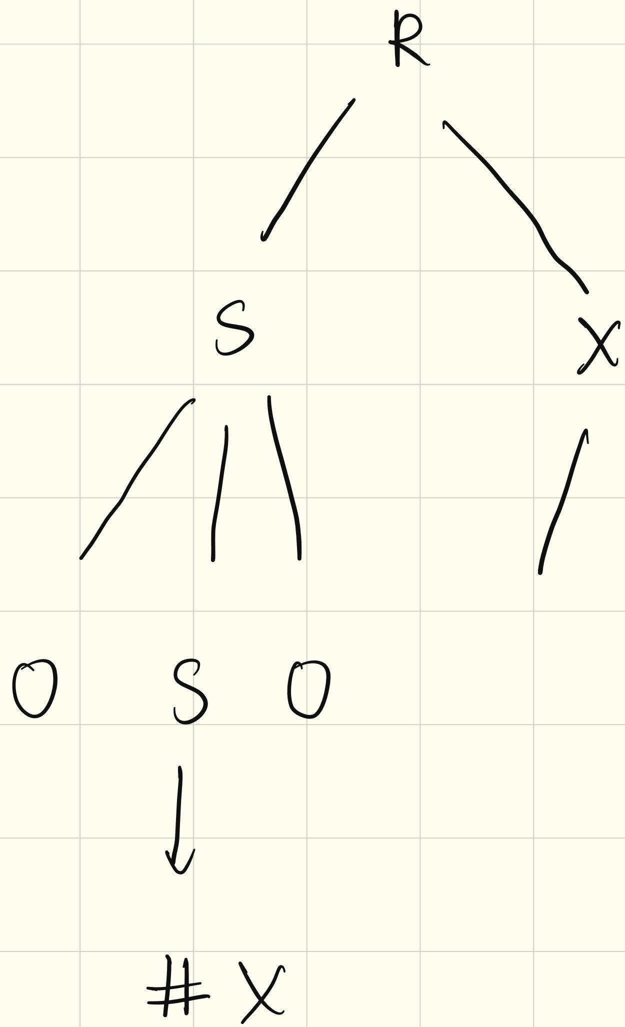
$T \to 0T0 \mid 1T1 \mid \# X$

$X \to$

0 0#1 0 0

$S \to TX$

$T \to 0S0 \mid$



$$S \rightarrow LTR$$

$$L \rightarrow \varepsilon \mid XM$$

$$R \rightarrow \varepsilon \mid MX$$

$$M \rightarrow \# \mid \# X M X \#$$

$$T \rightarrow 0T0 \mid 1T1 \mid 0M0 \mid 1M1 \mid \varepsilon$$

0 T 0

$\downarrow$

0 1 M 1 0

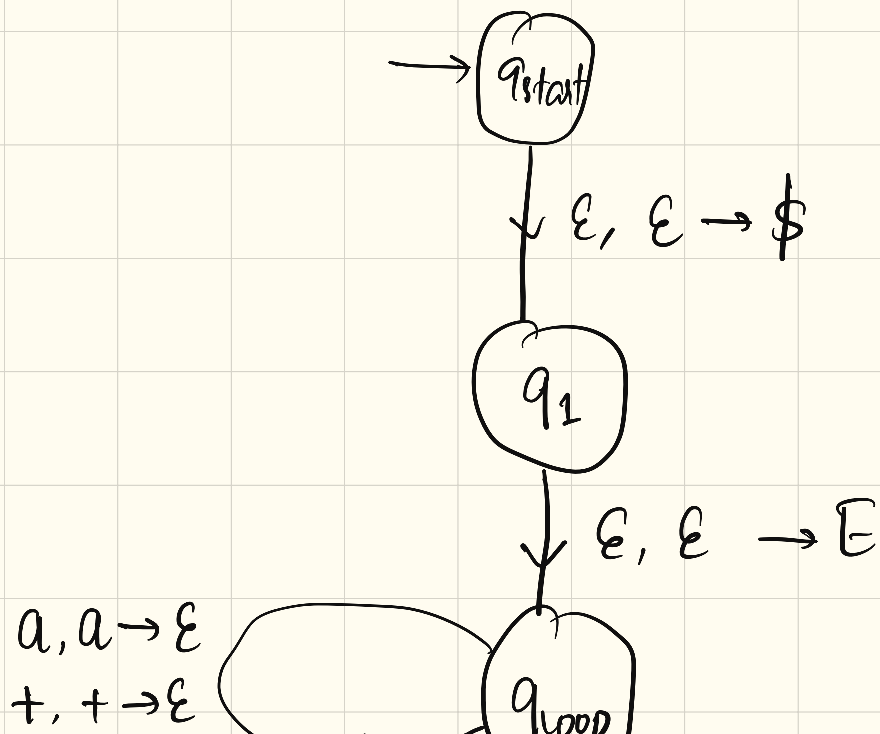
x M x

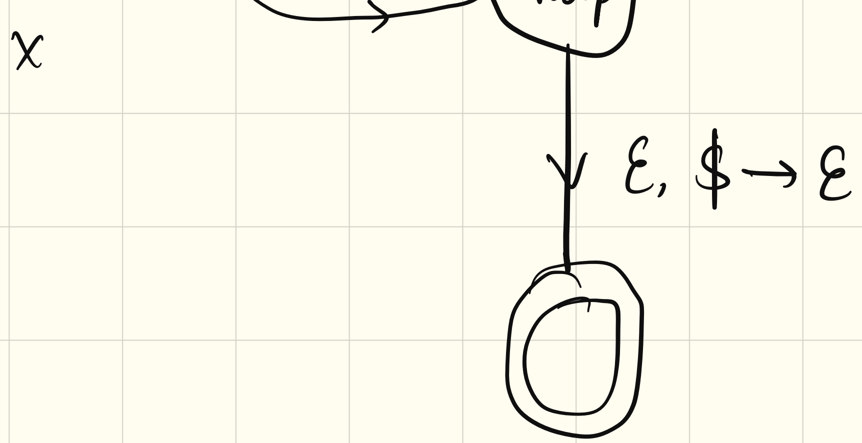
$$S \to S_1 \mid S_2$$

$$S_1 \to A S_1 B \mid C \mid \varepsilon$$

$a^n \ b^n$

$$C \to cC \mid \varepsilon$$





$A \to BAB \mid B \mid \varepsilon$

$B \to 00$

$A \to BAB \mid B \mid \varepsilon$

$B \to CC$

$C \to 0$

$A_0 \to A \mid \varepsilon$

$A \to BAB \mid B \mid \varepsilon$

$B \to CC$

$C \rightarrow 0$

$A_0 \rightarrow A \mid \varepsilon$

$A \rightarrow BAB \mid CC$

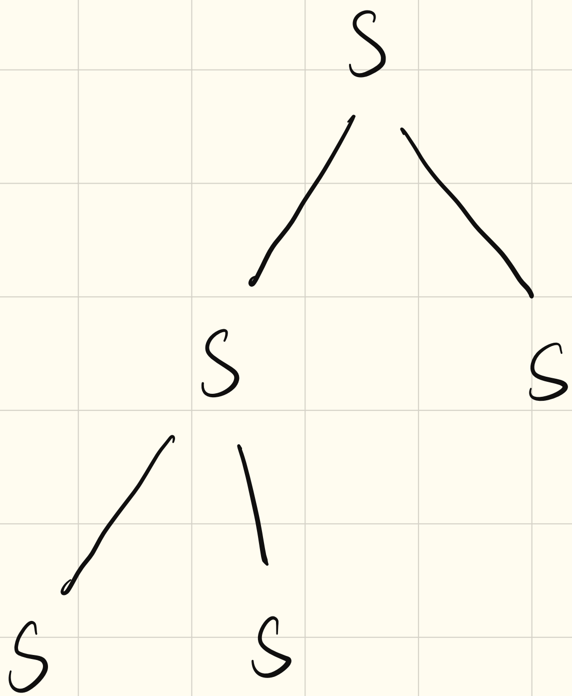
$B \rightarrow CC$

$C \rightarrow 0$

$A \rightarrow BA_1 \mid CC$

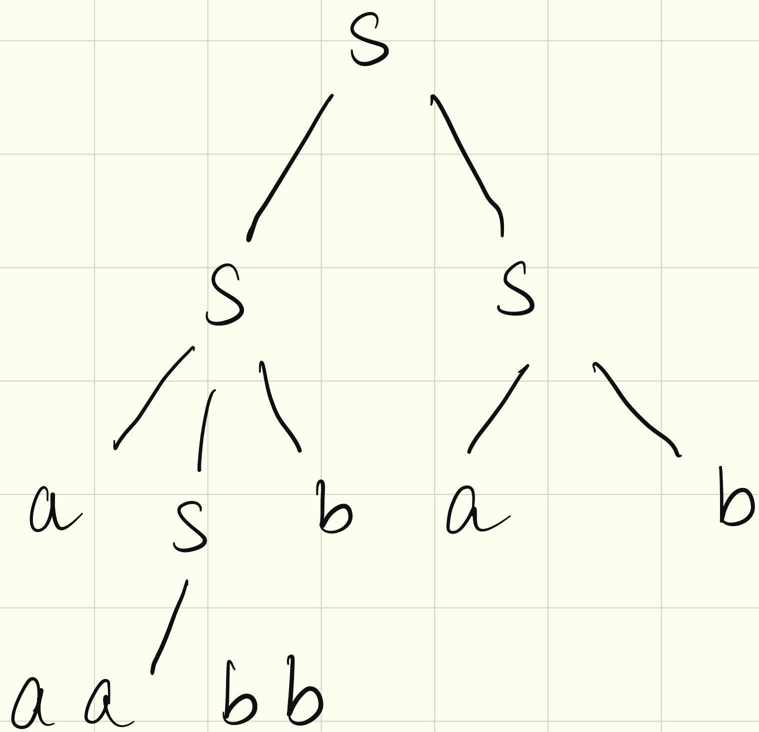
$A_1 \rightarrow AB$

$A_2 \rightarrow BA$



$S \to 0 | 1$

$0 \cup 1$



$$M = (Q_1, \Sigma, \Gamma, \delta_1, q_1, F_1) \rightarrow C$$

$$N = (Q_2, \Sigma, \delta_2, q_2, F_2)$$

$$P = (Q_1 \times Q_2, \Sigma, \Gamma, \delta, (q_1, q_2), F_1 \times F_2)$$

$$S \rightarrow A \# B \mid B \# A$$

$$B \rightarrow TBT \mid 1$$

$$A \rightarrow TAT \mid 0$$

$$T \rightarrow 0 \mid 1$$

$S \rightarrow AB \mid BC$

$A \rightarrow BA \mid a$

$B \rightarrow CC \mid b$

$C \rightarrow AB \mid a$