27 Jan 2025 – Theory of Computation – Week 04

DFA $\equiv$ reg-exp

$\equiv$ $\equiv$

NFA

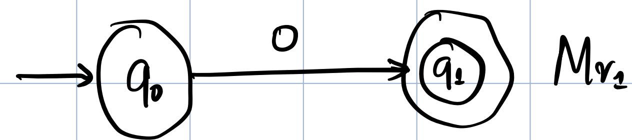
reg - exp $\longrightarrow$ NFA

closure under properties

DFA $\longrightarrow$ reg-exp

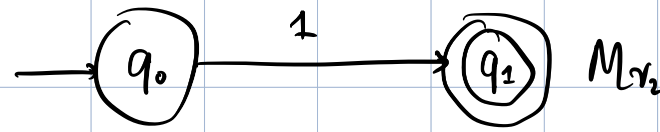
$(0 + 01)^*$

$r_1 = 0$



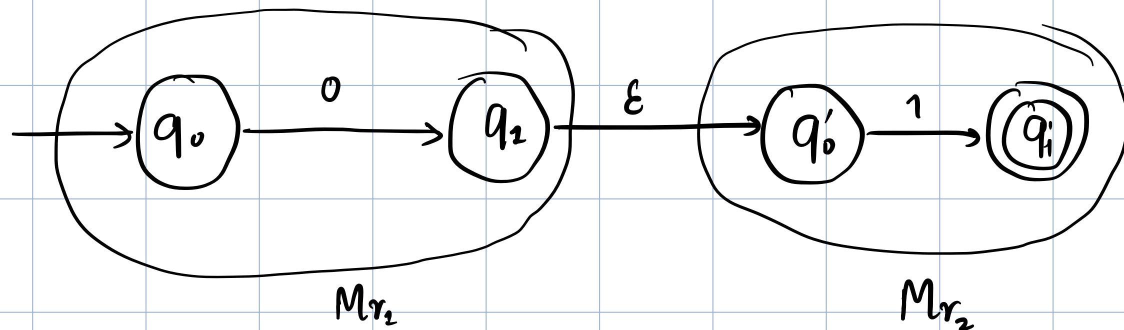
NFA

$r_2 = 1$

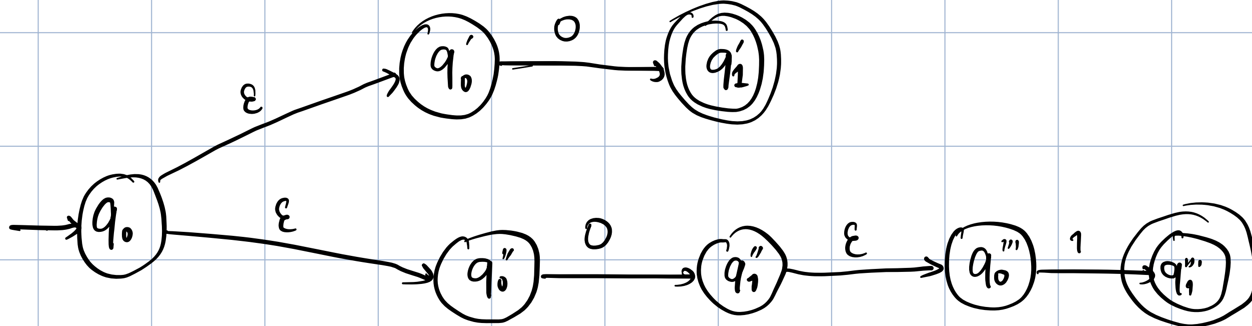


To construct NFA, we mimic our proof method

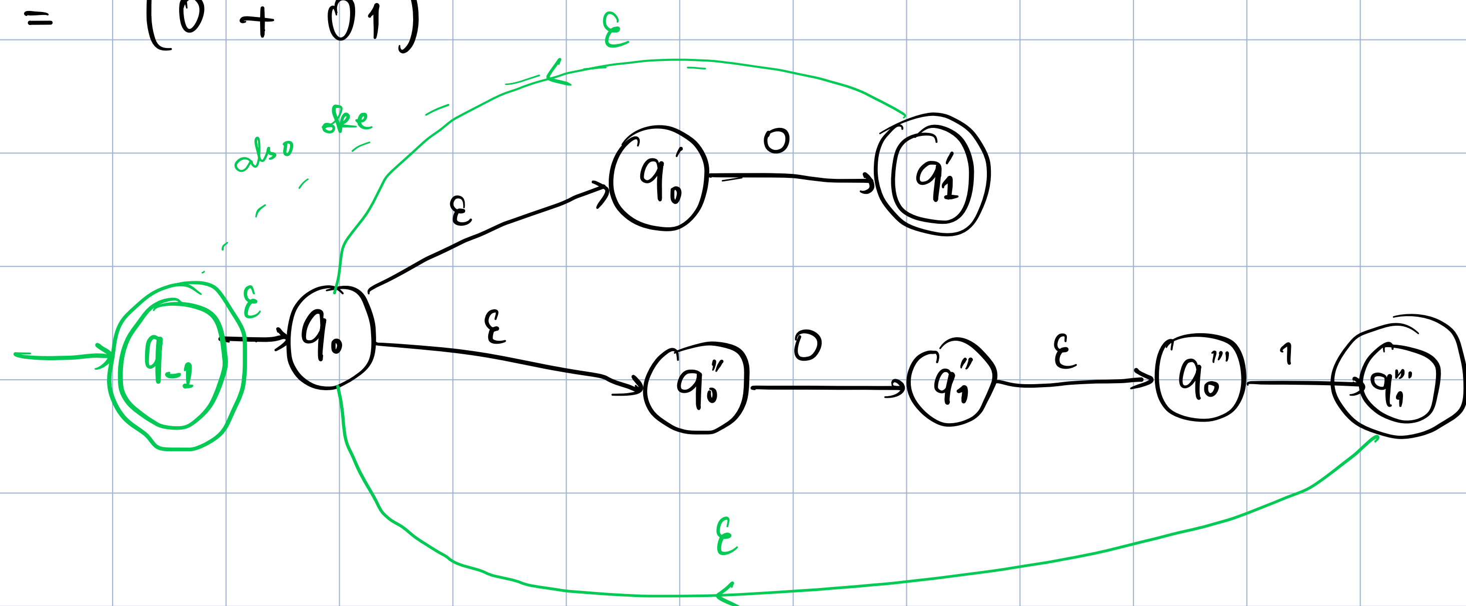
$r_3 = 01$



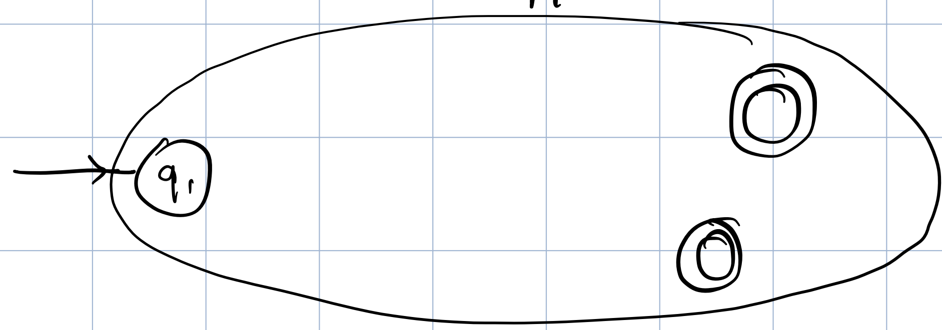
$r_4 = 0 + 01$



$r_5 = (0 + 01)^*$



DFA



$Q = \{q_1, \ldots, q_n\}$

for $i,j \in \{1,\dots,n\}$ and $k \in \{0, 1, \dots, n\}$

R_{ij}^{0}

$$\underbrace{q_i \xrightarrow{a,b} q_j}_{a+b}$$

$R_{ij}^{k} :=$ set of all strings x s.t

$$\hat{\delta}(q_0, x) = q_j$$

and passes through states numbered $\{q_1, \dots, q_k\}$

<u>prove by induction on k</u>

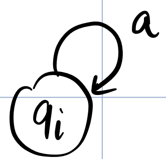
Base case:

R_{ij}^0

$$\underbrace{q_i \xrightarrow{a,b} q_j}_{a+b}$$

atomic regex

q_i $\qquad$ q_j $\qquad$ $R_{ij}^0 = \phi$



$i = j$

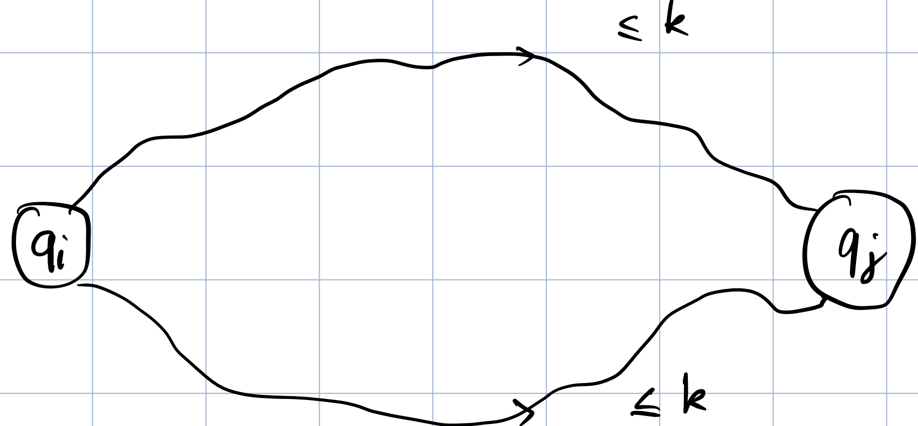
$$R_{ii}^0 = \{a, \varepsilon\}$$

$$R_{ii}^0 \neq a^*$$

$$q_i \xrightarrow{a} q_i \xrightarrow{a} q_i$$

Induction:

R_{ij}^k

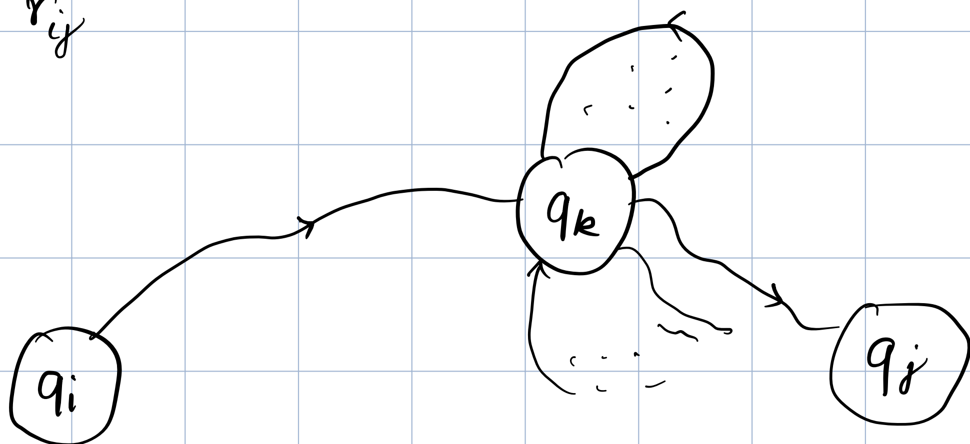


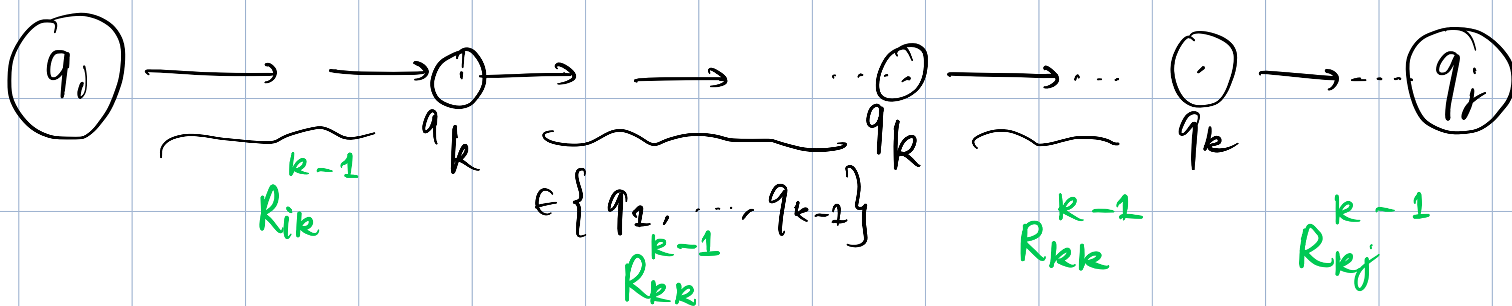
R_{ij}^{k} $\xrightarrow{\text{Case 1}}$ paths that don't touch the states q_k at all
$\{q_1, \dots, q_{k-1}\}$

R_{ij}^{k} $\xrightarrow{\text{Case 1}}$ paths that passes through q_k

Case 1 : $R_{ij}^{k-1} \equiv r_{ij}^{k-1}$

Case 2 :





regex for all strings in case 2 $\quad R_{ik}^{k-1} \cdot \left(R_{kk}^{k-1}\right)^* \cdot R_{kj}^{k-1}$

by induction on k

$$r_{ik}^{k-1} \cdot \left(r_{kk}^{k-1}\right)^* \cdot r_{kj}^{k-1}$$

$$\text{case 1} + \text{case 2} = r_{ij}^{k-1} + r_{ik}^{k-1} \left(r_{kk}^{k-1}\right)^* \cdot r_{kj}^{k-1}$$

R_{1j}^{n} := set of all strings that takes you from $q_1 \longrightarrow q_j$

$$L(M) = \bigcup_{f \in F} R_{ij}^{n}$$

say $F = \{ j_1, j_2, j_3 \}$ $\quad L(M) = r_{1j_1}^{n} + r_{1j_2}^{n} + r_{1j_3}^{n}$

no. of regex $= 4n^3$

29 Jan 2025

→ Exam syllabus : up-to last lecture.

Are all sets regular sets?

Are all finite sets regular sets?

$$L_2 = \{ a^n b^n \mid n \geq 1 \}$$

→ Finite automata has finite memory.

→ We need to remember no. of a's in the input and check with b's

→ Equivalence testing requires more memory with $n \uparrow$

If $L_1' = \{ a^n b^n \mid 100 \geq n \geq 1 \}$

$$|L_1| \leq 100$$

→ Any language of finite size is regular → construct DFA by brute force.

$L_2 = \{ x \mid x \in \{0, 1\}^* $ s.t. x has equal no. of 01 and 10 as substrings

$\boxed{\text{Regular}}$

$0110 \in L_2$ $\qquad$ $0101 \notin L_2$

→ there can be an easier way to describe a finite automata for a language.

Pumping Lemma

A is regular $\Longrightarrow$ A has property $\mathcal{P}$

if B does not have the property $\mathcal{P}$, B is not regular

There are non-regular languages which have $\mathcal{P}$

→ necessary but not sufficient

Some large const

If A is regular, then $\exists$ a number t such that for all $x \in A$ and $|x| \geq t$ $\exists$ u, v, w s.t. $x = uvw$

string of large length

and the following holds:

(1) $u(v)^i w \in A$ $\quad \forall i \geq 0$

(2) $v \neq \varepsilon$ → pump middle part any no. of times

(3) $|uv| \leq t$

loop

$u\,(v)^i\,w$ $\quad$ $i = 0$, $uw \in A$

$x = x_1 \; x_2 \; \cdots \; x_n$

Let M be a DFA recognizing A

$Q = \{ q_0, q_1, \ldots, q_e \}$

$F = \{ q_{30}, \ldots \}$

$$q_0 \xrightarrow{x_1} q_2 \xrightarrow{x_2} q_{19} \xrightarrow{x_3} q_{25} \cdots \xrightarrow{x_n} q_{30}$$

→ No. of states in the run of x $= n + 1$

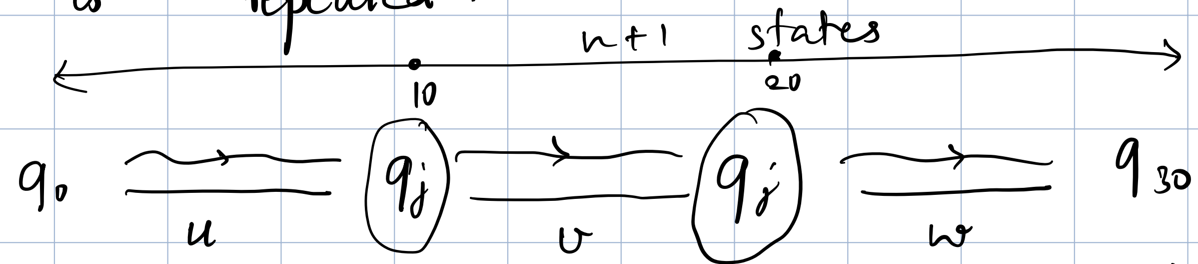
If $n + 1 \geq |Q| + 1$

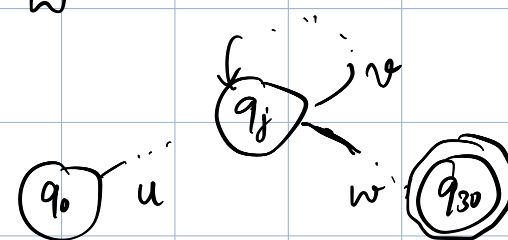
or $n \geq |Q|$

at least one state must be repeated

$|u| + |v| \leq n$

Suppose q_j is repeated :





$x = u v w$

$x \in A \implies u v^i w \in A \quad \forall\ i \geq 0$

$v = \varepsilon$?

$\rightsquigarrow$ v cannot be empty.

because no. of steps was assumed to be $\geq |Q| + 1$

Pigeonhole principle

10^{th} step $\qquad$ 20^{th} step

q_i $\qquad$ q_j

$$q_0 = q_0' \xrightarrow{x_1} \cdots \overset{q_s}{\overset{\parallel}{q_i'}} \underbrace{\xrightarrow{x_2} q_2' \longrightarrow \cdots}_{v \neq \varepsilon} \overset{q_s}{\overset{\parallel}{q_k'}} \longrightarrow \cdots \longrightarrow q_n'$$

where $q_i \in \{ q_0, \ldots, q_\ell \}$

and $n \geq \ell + 1$

$\underset{\substack{j \neq k \\ k > j}}{\exists}$ q_j', q_k' s.t $q_j' = q_k' = \overset{\text{some state}}{q_s}$

$\therefore \quad k - j \geqslant 1$

$\downarrow$

??? } Pigeonhole principle

$\downarrow$

$v \neq \varepsilon$

$\rightarrow$ t should be $|Q|$ (finite)

$x \in A$ and $|x| \geqslant t$

If there are no strings of length $\geqslant t$

then the lemma is vacuously true

$L_1 = \{ 0^n 1^n \mid n \geq 1 \}$

Proof by contradiction:

Assume L_1 is regular. Then,

$\exists\ t$ s.t. properties hold.

Consider $x = 0^t 1^t$ $\quad |x| = 2t > t$

Then, $\exists\ u, \underset{\neq \varepsilon}{v}, w \in \Sigma^*$ s.t.

$x = uvw$ and $|uv| \leq t$

then $uvvw \in A$

$\Rightarrow \Leftarrow$

v must be within the first t ones

30 Jan 2025

Pumping Lemma: Let A be regular, then $\exists$ a natural number t such that

$\forall\ x \in A$ and $|x| \geq t$, $\exists\ u, v, w \in \Sigma^*$ such that $x = uvw$ and the following holds

(i) $v \neq \varepsilon$

(ii) $uv^iw \in A \quad \forall\ i \geq 0$

(iii) $|uv| \leq t$

If this property does not hold for a language, it is not regular

Example 1: $A = \{ 0^n 1^n \mid n \geq 0 \}$

Assume that A is regular, then

Pumping lemma $\Rightarrow \exists\ t$ s.t $\mathcal{P}$ holds

General methodology:

* find one $x \in A$ and $|x| \geq t$ must work for any t

Prove that for all x with length $> t$, there exists u, v, w s.t $x = uvw$ but $uv^i w \notin A$

* Consider all possible ways of breaking x into $u, v, w \in \Sigma^*$ such that $x = uvw$, $v \neq \varepsilon$, $|uv| \leq t$

* find j such that $uv^j w \notin A$.

Consider : $x = 0^t 1^t$

$v \neq \varepsilon$ and $|v| \leq t$

v = non-zero length string of 0

$= 0^{\lambda}$ where $\lambda > 0$

Holds true in all possible ways of breaking x

Consider $uvvw$

$0^{t+\lambda} 1^t$

$B = \{ xx \mid x \in \{0,1\}^* \}$

$y \in B, \quad |y| \geq t$

$y = 0^t 0^t$ ✗ Bad choice

can be always pumped

$u = 0^{t-2}$

$v = 00$

$w = 0^t$

$y = 0^t 1 \; 0^t 1$

u

$v = 0^n$ $\because |v| \leq t+1$

w

$\rightarrow uvvw \notin A.$

$$C = \{ 0^{n^2} \mid n \geq 0 \}$$

Show that C is not regular.

$\rightarrow$ Suppose C is regular. Pumping lemma gives you t

Consider $x = 0^{t^2} \in C$

$|x| \geq t$

Consider any valid way of breaking $x = uvw$ $\quad v \neq \varepsilon$

$|v| \leq t$

$$t^2 + 1 \leq |uvvw| \leq t^2 + t$$

$$\Rightarrow \quad t^2 < |uvvw| < (t+1)^2 \quad \Rightarrow \Leftarrow$$

$D = \{ 0^n 1^m \mid n \geq m \}$

Suppose D is regular. By Pumping lemma $\exists\ t$ such that $\mathcal{P}$ holds.

Consider $x = 0^t 1^t = uvw$

$v \neq \varepsilon \qquad |v| \leq t$

$uv^iw \in D$

$i = 0 \qquad uw \qquad \Rightarrow\Leftarrow$

$E = \{ x \in \{0, 1\}^* \mid x$ has equal number of 0's and 1's $\}$

Suppose E is regular.

$$E \cap 0^* 1^* = A$$