20 Jan 2025 - Theory of Computation - Week 03

NFA

Thm: Let N be a NFA. Then the language recognized by N is a regular language

$L(N)$

Proof: $N = (Q, \Sigma, \delta, q_0, F)$

Simplifying assumption :- no ε transition

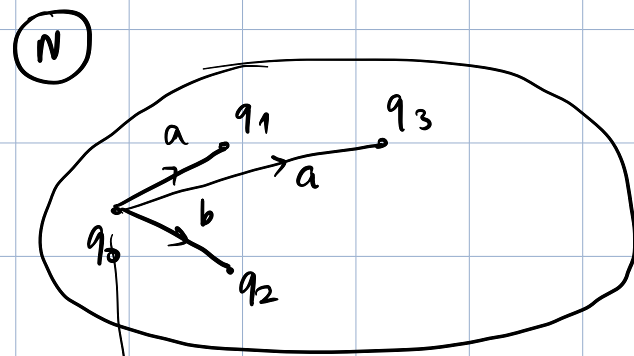
Construct a DFA M such that

$$L(M) = L(N)$$

$$M = (Q', \Sigma, \delta, \{q_0\}, F')$$

$$Q' = \mathcal{P}(Q) = \{ S \mid S \subseteq Q \}$$

$$\delta'(\{q_0\}, a) = \{q_1, q_3\}$$



$$\underset{\forall S \subseteq Q}{\delta'(S, a)} = \bigcup_{q \in S} \delta(q, a)$$

$$\{q_0\} \xrightarrow{a} \{q_1, q_3\}$$

$$F' = \{ S \subseteq Q \mid \underbrace{S \cap F \neq \phi}_{\text{any one element in } S \text{ should be a final state.}} \}$$

$\forall x \in \Sigma^*$,

$\hat{\delta}(q_0, x) :=$ set of all possible states N can be after reading x

$\hat{\delta}_N(q_0, xa) = \bigcup_{q \in \hat{\delta}_N(q_0, x)} \delta_N(q_0, a)$

$\hat{\delta}_N(q, \varepsilon) = q$

$x \in L(N) \iff \hat{\delta}(q_0, x) \cap F \neq \phi$

Claim: $\forall x \in \Sigma^*$

$$\hat{\delta}_N(q_0, x) = \hat{\delta}'_M(\{q_0\}, x)$$

Proof: By induction on the length of x, $|x|$

Base case: $|x| = 0$, $x = \varepsilon$

assume no ε transition

$\delta_N - \{q_0\}$

$\delta_M \to \{q_0\}$

I.H: true for all $|x| \leq l$

Consider $xa \in \Sigma^*$

To prove

$\hat{\delta}_N(q_0, xa)$ $\quad$ $\hat{\delta}'_M(\{q_0\}, xa)$

$= \hat{\delta}'_M(\hat{\delta}'_M(\{q_0\}, x), a)$

$$\hat{\delta}'_M(\{q_0\}, xa)$$

$$= \hat{\delta}'_M(\hat{\delta}'_M(\{q_0\}, x), a) \qquad \text{By def}^n$$

$$= \hat{\delta}'_M(\hat{\delta}_N(q_0, x), a) \qquad \text{by I.H}$$

$$= \bigcup_{q \in \hat{\delta}_N(q_0, x)} \delta_N(q, a) \qquad \text{by def}^n$$

by def^n

$$= \hat{\delta}_N(q_0, xa)$$

- Complete proof : $L(M) = L(N)$

Claim 2 :

$$x \in L(N) \iff x \in L(M)$$

$$x \in L(N) \iff \hat{\delta}_N(q_0, x) \cap F \neq \phi \longrightarrow 1$$

$$\Updownarrow$$

$$\hat{\delta}'_M(\{q_0\}, x) \in F' \longrightarrow 2$$

$$\Updownarrow$$

$$x \in L(M)$$

From 1 and 2

$$L(M) = L(N)$$

How to handle ε transition

Non-determinism ⟶ non-dertiministic choices } → set of choices

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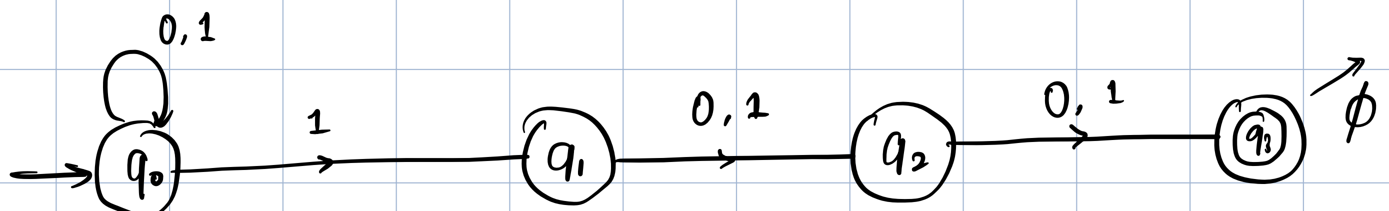
Example: all strings that have 1 at 3rd place from last
right



am I
at $(n-2)^{th}$
input of
the string ?

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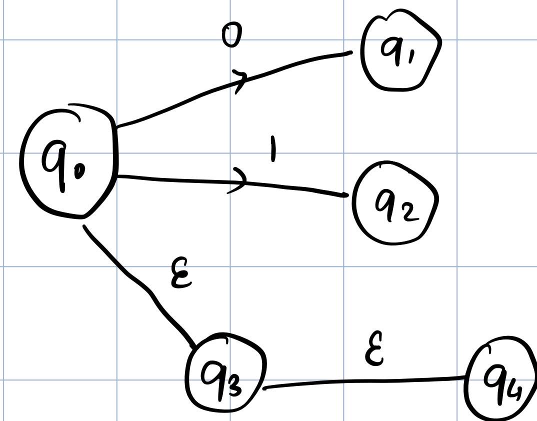
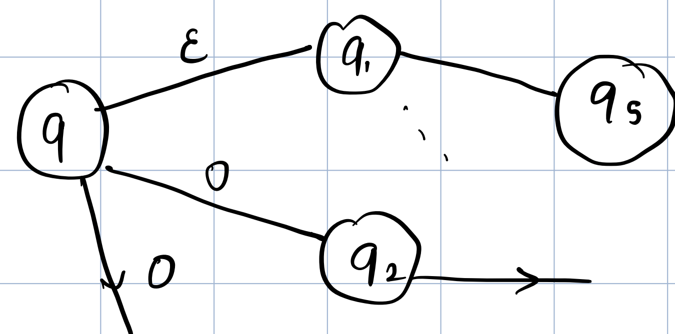


$$\{q_0\} \cup \hat{\delta}(q_0, \varepsilon)$$

$$\hat{\delta}(q, x) = \hat{\delta}(q_1, x) \cup \hat{\delta}(q_5, x) \cup$$

all vertices reachable only using ε transitions



for any state q,

$\varepsilon - cl(q)$ = # of set of states reachable from q among only ε-transition.

$$\hat{\delta}(q, \varepsilon) = \varepsilon\text{-cl}(q)$$

$$\hat{\delta}(q, xa) \qquad\qquad a \in \Sigma$$

$$= \varepsilon\text{-cl}\left(\{\delta(r, a) \mid r \in \hat{\delta}(q, x)\}\right)$$

$$= \varepsilon\text{-cl}\left(\bigcup_{r \in \hat{\delta}(q,x)} \delta(r, a)\right)$$

<u>Recap:</u>

- Regular sets/ languages $\rightleftarrows$ regular expressions

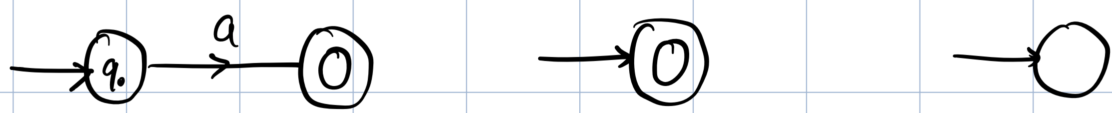
  $\parallel$
- DFA

  $\updownarrow \; \parallel$

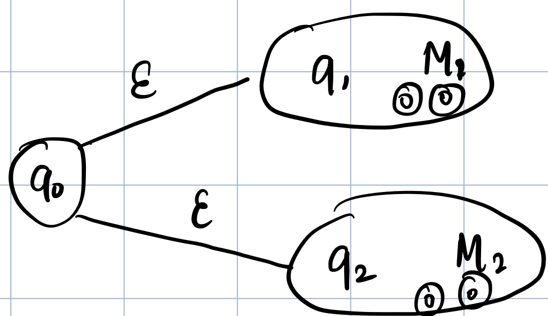
  NFA

Thm: Let r be a regular expression. Then $L(r)$ is regular.

Proof: $a \in \Sigma$, ε, ϕ

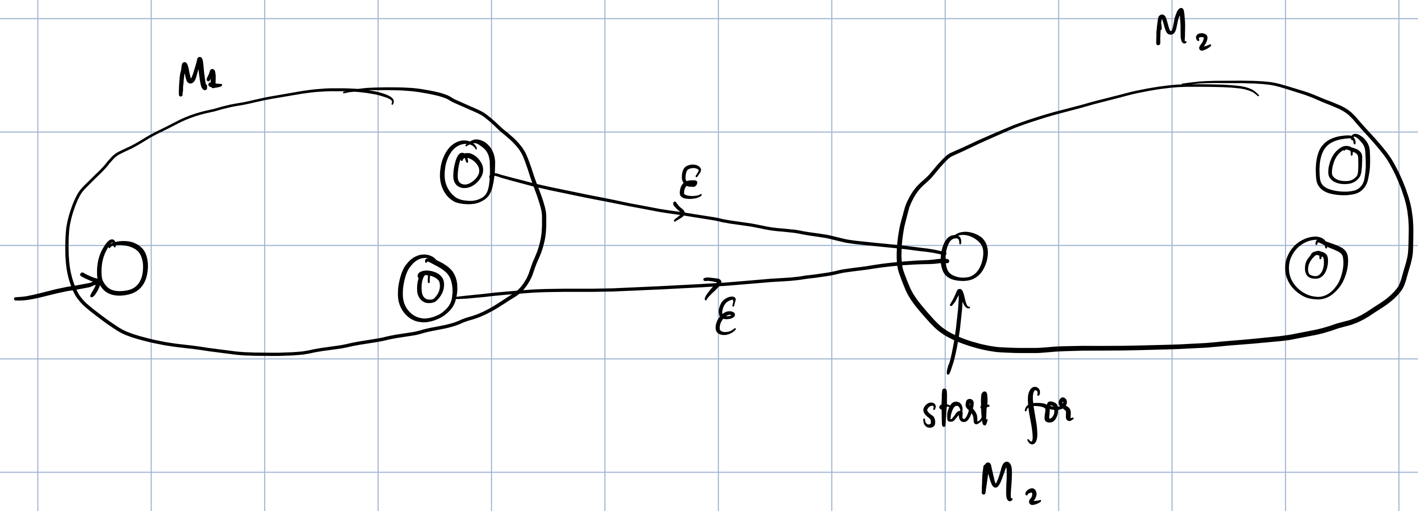


Union:



Concatenation

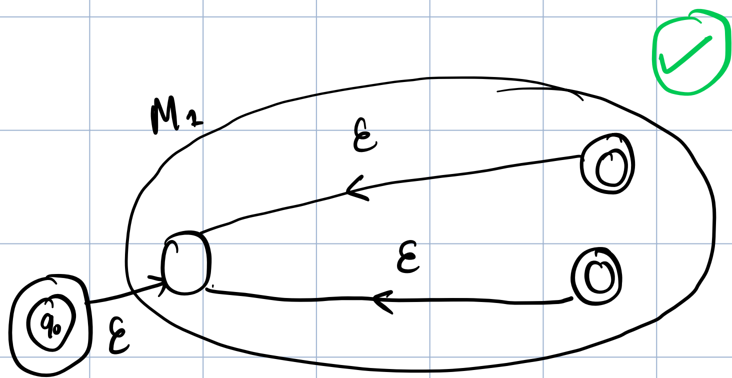
$$L_1 \cdot L_2 = \{ x \mid x = yz \text{ when } y \in L_1, z \in L_2 \}$$

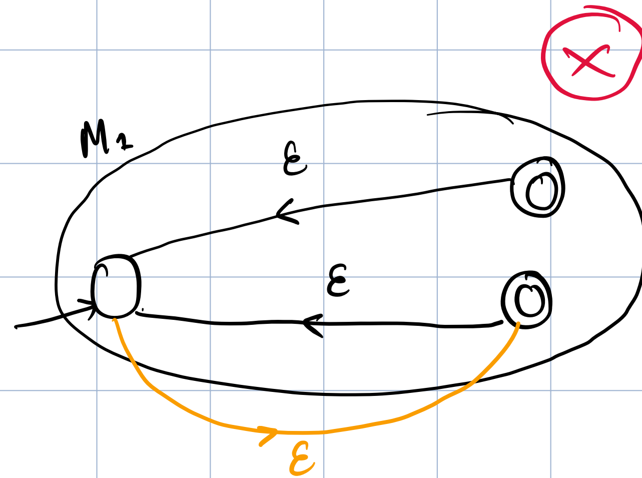


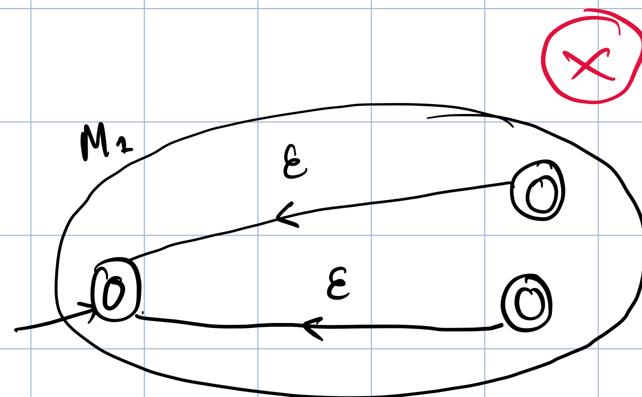
o Proof

Star

$$(L)^* = \{ x \mid x = x_1 x_2 \dots x_k,\ k \geq 0 \text{ each } x_i \in L \}$$







DFA $\longrightarrow$ regular expressions

$L_1, M_1 \longrightarrow r$ over Σ s.t. $L(r) = L_1$

<u>Proof</u>: $M_1 = (Q, \Sigma, \delta, q_1, F)$

$Q = \{q_1, \dots, q_k\} = \{1, \dots, k\}$

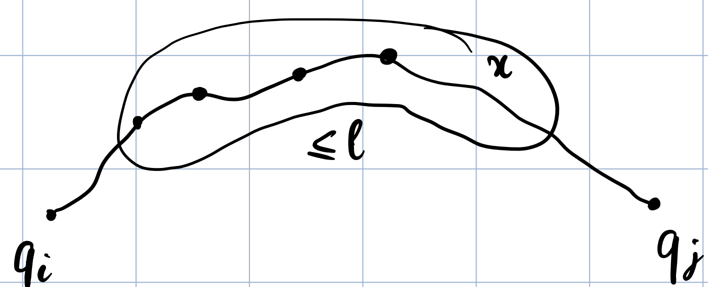
R_{ij}^{ℓ} := set of strings s.t $\hat{\delta}(q_i, x) = q_j$ and it pass through only states in $\{1, \dots, \ell\}$

all strings that take n from i to j only through $1, \dots, \ell$

$0 \leq \ell \leq k$

$1 \leq i, j \leq k$

R_{ij}^{K} = set of all strings x such that $\hat{\delta}(q_i, x) = q_j$.



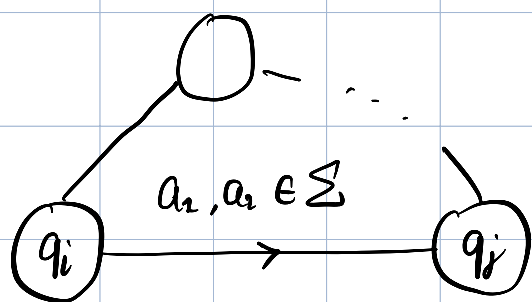
Let $f \in F \subseteq \{1, \ldots, k\}$

regular operation

$$\bigcup_{f \in L} R^{k}_{1f} = L_1 = L(M_1)$$

Note: we working with DFA

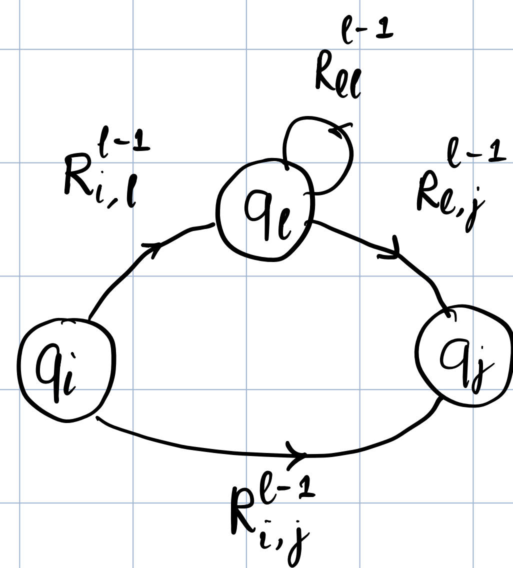
If $l = 0$



$$R^{0}_{ij} = \{a_1, a_2\}$$

$$\Downarrow$$

$$a_1 + a_2$$



$$R^{l}_{ij} = R^{l-1}_{il} \cdot \left(R^{l-1}_{ll}\right)^{*} \cdot R^{l-1}_{lj}$$