19 Nov 2024 - Linear Algebra - Week 16

Proof of Schur triangularization:

$A \in M_n(\mathbb{C})$ | Claim $Q^* Q = Q Q^* = I$

and $Q^* A Q = T$, T is upper triangular

$\chi_A(t) \longrightarrow$ it has all its roots in $\mathbb{C}$

$\hookrightarrow$ A has eigenvalues as the roots of $\chi_A(t) = 0$

Let λ_1 be an eigenvalue of A

i.e., $q_1 \neq 0$ s.t. $A(q_1) = \lambda_1 q_1$, $q_1^* q_1 = 1$

Consider an orthonormal basis of $M_{n\times 1}(\mathbb{C})$

$\{q_1, \ldots, q_n\} \longrightarrow$ Basis

- ⊛ extension of basis
- ⊛ Gram-Schmidt process

$$Q_1 = [\, q_1 \mid q_2 \mid \cdots \mid q_n \,]$$

$$q_i^* q_j = \delta_{ij} \iff Q_1^* Q_1 = I = Q_1 Q_1^*$$

$$Q_1^* A Q_1 = Q_1^* [\, A(q_1) \mid A(q_2) \mid \cdots \mid A(q_n) \,]$$

$$= Q_1^* [\, \lambda q_1 \mid \cdots \quad]$$

$$= \begin{bmatrix} q_1^* \\ \vdots \\ q_n^* \end{bmatrix} \cdot [\, \lambda q_1 \mid \cdots \quad]$$

$$= [\, \lambda_1 e_1 \mid Q_1^* A q_2 \mid Q_1^* A q_3 \mid \cdots \mid Q_1^* A q_n \,]$$

$$= \left[\begin{array}{c|c} \lambda & \cdots \\ \hline 0 & \\ \vdots & (A_2)_{(n-1)\times(n-1)} \\ 0 & \end{array}\right]$$

We will prove Schur's decomposition using induction.

Every matrix is triangularizable

Base case: $M_2(\mathbb{C})$ is triangularizable

$B \in M_2(\mathbb{C})$

Let λ be eigenvalue ; $q_1 \neq 0$: $B(q_1) = \lambda_1 q_1$, $\underbrace{q_1^* q_1 = 1}_{\text{how?}}$??

Take $q_2 \neq 0$ and $q_2^* q_1 = 0$, $q_2^* q$

$$Q = [q_1 | q_2]$$

$$Q^* B Q = \begin{bmatrix} \lambda_1 & \# \\ 0 & \# \end{bmatrix}$$

A_2 can be triangularized i.e.,

$Q_2 \in M_{n-1}(\mathbb{C})$, $Q_2^* Q_2 = I = Q_2 Q_2^*$

$Q_2^* A_2 Q_2 = T_2$, T_2 is upper triangular

$$Q_0 = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & A_2 \end{array}\right]$$

$$Q_0^* Q_0 = I_n$$

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q^* \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

$$Q_1^* A Q_1 = \left[\begin{array}{c|c} \lambda_1 & \cdots \\ \hline 0 & A_2 \end{array}\right]$$

$$Q_2^* A Q_2 = T_2$$

$$A = Q_2 T_2 Q_2^*$$

$$= \left[\begin{array}{c|c} \lambda_1 & \cdots \\ \hline 0 & Q_2 T_2 Q_2^* \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} \left[\begin{array}{c|c} \lambda_1 & \cdots \\ \hline 0 & T_2 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q_2 \end{array}\right]^*$$

$$= Q_0 \left[\begin{array}{c|c} \lambda_1 & \cdots \\ \hline 0 & T_2 \end{array}\right] Q_0^*$$

$$\Rightarrow Q_1^* A Q_1 = Q_0 T Q_0^*$$

$$\Rightarrow Q_0^* Q_1^* A Q_1 Q_0 = T$$

Singular - value decomposition

→ Polynomial Functional Calculus

→ Cayley - Hamilton Theorem

$A \in M_{m \times n}(\mathbb{C})$ not identically zero.

$$\text{rank}(A) = r > 0$$

$\exists$ Unitary $U \in M_m(\mathbb{C})$

$V \in M_n(\mathbb{C})$

and

$$\Sigma = \left[\begin{array}{cccc|ccc} \sigma_1 & 0 & \cdots & & 0 & \cdots & 0 \\ 0 & \sigma_2 & & & \vdots & & \vdots \\ \vdots & & \ddots & & \vdots & & \vdots \\ \vdots & & & \sigma_r & \vdots & & \vdots \\ \hline & & 0 & & & 0 & \end{array}\right]$$

$$A_{m\times n} = U_{m\times m} \Sigma_{m\times n} V^*_{n\times n}$$

σ_i is called singular value of A.

Singular value: Let $\{\lambda_1, \dots, \lambda_r\}$ be the non-zero eigenvalues of $A^* A$.

$A^* A$ is positive matrix.

$\forall X \in M_{n\times 1}(\mathbb{C})$

$X^* (A^* A) X \geq 0$ } verify

So $A^* A$ is a positive matrix

$\Rightarrow \lambda_1, \dots, \lambda_r$ are non-negative

Problem: $\text{Rank}(A^* A)$

$= \text{rank}(A^*)$

$= \text{rank}(A)$

A^*A has $(\text{rank}(A^*A))$ many positive eigenvalues ??

Define $\sigma_i = \sqrt{\lambda_i}$; $i = 1, \dots, r$

σ_i's are called singular values of A.

$A \in M_{m\times n}(\mathbb{C})$, $U \in M_{m\times n}(\mathbb{C})$, $V \in M_{n\times n}(\mathbb{C})$

and

$$\Sigma = \left[\begin{array}{cccc|ccc} \sigma_1 & 0 & \cdots & & 0 & \cdots & 0 \\ 0 & \sigma_2 & & & \vdots & & \vdots \\ \vdots & & \ddots & & & & \\ \vdots & & & \sigma_r & \vdots & & \vdots \\ \hline & & \mathbf{0} & & & 0 & \end{array}\right]$$

$$\boxed{A_{m\times n} = U_{m\times m} \Sigma_{m\times n} V^*_{n\times n}} \text{ — SVD}$$

- Use diagonalization of $A^* A$
- Use $\sigma_i = \sqrt{\lambda_i}$

Polynomial Functional Calculus

Let $A \in M_n(\mathbb{C})$ and p be a (non-constant) polynomial, then λ is an eigenvalue of A if and only if $p(\lambda)$ is an eigenvalue of $p(A)$.

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_2 t^d$$

$$p(A) = a_0 I + a_1 A + \cdots + a_2 (A^d)$$

$$A = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}$$

$$p(x) = x^2 + 2x + 1$$

$$p(A) = \begin{bmatrix} p(\alpha) & (*) \\ 0 & p(\gamma) \end{bmatrix}$$

If T is triangular then

$$[p(T)]_{ij} = p([T]_{ii}) \qquad i = 1, \dots, n$$

$$\chi_A(t) = \chi_T(t)$$

$$T = QAQ^* \qquad (\text{by Schur})$$

$$p(Q^* A Q)$$

$$= (Q^* A Q)^2 + 2Q^* A Q + I$$

$$= Q^* A Q Q^* A Q + 2Q^* A Q + Q^* Q$$

$$= Q^* (p(A)) Q$$

$$\chi_{p(A)}(t) = \chi_{p(T)}$$

$$p(T) = p(Q^* A Q) = Q^* (p(A)) Q$$

What are the roots of $\chi_{p(T)}(t) = 0$

diagonal for $p(T)$, $p([T]_{ii})$ $i = 1, \dots, n$

If λ is an eigenvalue for A, then $p(\lambda)$ is an eigenvalue

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- Polynomial functional calculus
- Cayley – Hamilton
- SVD – Singular Value decomposition ($A = U \Sigma V^*$)

Polynomial functional calculus

Let $A \in M_n(\mathbb{C})$ and let $p \in \mathbb{C}[z]$ be a polynomial. Then λ is an eigenvalue for A iff $p(\lambda)$ is an eigenvalue for $p(A)$.

Proof: Use schur triangularization.

$$U^* A U = T \quad , \quad T \to \text{upper triangular}$$

$$\chi_A(t) = \chi_{U^*AU}(t) = \chi_T(t) \longrightarrow \text{(i)}$$

and roots $\chi_{p(T)}(t)$ are $[p(T)]_{ii}$ $\longrightarrow$ (ii)

$$[p[T]]_{ii} = p([T]_{ii}) \longrightarrow \text{(iii)}$$

Combine (i), (ii), (iii) and the fact that T

$$A = \begin{bmatrix} c_0 & c_2 & c_1 \\ c_1 & c_0 & c_2 \\ c_2 & c_1 & c_0 \end{bmatrix}_{3\times 3} \rightsquigarrow \text{permutation involved}$$

↳ eigenvalue → easy

what if A was 100×100 ?

$$c_0 I + c_1 \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{P} + c_2 \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P^2}$$

→ permutation matrices ←

P : $e_1 \to e_2 \to e_3 \to e_1$

P^2 : $e_1 \to e_3 \to e_2 \to e_1$

$$A = c_0 I + c_1 P + c_2 P^2$$

Eigenvalues for A are eigenvalues for P evaluated at

$$q(t) = c_0 + c_1 t + c_2 t^2$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -t & 0 & 1 \\ 1 & -t & 0 \\ 0 & 0 & -t \end{vmatrix} \qquad -t\left(t^2\right) + 1 = 1 - t^3$$

roots: $1,\ e^{i2\pi/3},\ e^{i4\pi/3}$

eigenvalues for P are $1, e^{\frac{i2\pi}{3}}, e^{\frac{i4\pi}{3}}$

$1, \omega, \omega^2$

eigenvalues for A

$$q(1) = c_0 + c_1 + c_2$$

$$q(e^{i2\pi/3}) = c_0 + c_1 e^{i2\pi/3} + c_2 e^{i2\cdot 2\pi/3}$$

$$= c_0 + c_1 \omega + c_2 \omega^2$$

$$q(e^{i4\pi/3}) = c_0 + c_1 \omega^2 + c_2 \omega$$

calculus $\rightarrow$ change

(*) SVD

$A \in M_{m \times n}(\mathbb{C})$, with singular value $\sigma_1, \ldots, \sigma_{r}$, rank(A)

$$\sigma_i = \sqrt{\lambda_i}, \quad \lambda_i \text{ is eigenvalue of } A^* A.$$

$i = 1, 2, \ldots, r$

$r = \text{rank}(A)$

Then, there exists U, V unitary s.t.

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V^*_{n \times n}$$

$(A^* A)_{n \times n}$ → square, positive, hermitian

$A^* A$ be unitarily diagonalized → eigenvectors are $\perp^{\text{lar}}$

$\lambda_1, \dots, \lambda_n$ be eigenvalues of $A^* A$, $\lambda_i \geq 0$, $i = 1, 2, \dots, n$

$V_1, \dots, V_n$ eigenvector $A^* A$ corresponding to $\lambda_1 \dots \lambda_r$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$

are mutually orthogonal, $v_i^* v_i = 1$

$$V = [V_1 | \dots | V_n]$$

$$V^* V = V V^* = I \quad \text{and} \quad V^* (A^* A) V = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\frac{A(v_i)}{\sigma_i} = u_i \qquad i = 1 \dots r$$

$$u_j^* \cdot u_i = \frac{v_j^* A^* A v_i}{\sigma_i \sigma_j} = \lambda_i \frac{v_j^* v_i}{\sigma_j \sigma_i}$$

$$u_j^* u_i = \delta_{ij}$$

$\{u_1, \ldots, u_r\}$ → mutually orthogonal in $M_{m\times 1}(\mathbb{C})$

↳ can be extendended to an orthonormal basis of $M_{m\times 1}(\mathbb{C})$

$\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_m\}$ o.n.b for $M_{m\times 1}(\mathbb{C})$

$$U = [u_1 | \ldots | u_m]_{m\times m}$$

$$A(v_i) = \sigma_i u_i \quad i = 1, \ldots, r$$

$$A(V) = \underset{m\times m}{U} \underset{m\times n}{\Sigma}$$

$$\begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & 0 & & \\ 0 & & \sigma_r & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}_{m\times n}$$

$\chi \in \mathbb{C}[z]$, $p(t) = \det(A - tI)$, $A \in M_n(\mathbb{C})$

↘ non constant polynomial

C. H

$$\chi(A) = 0$$

Euclidean algorithm

given p, q then

$$p = h \cdot q + r$$

where $\deg(r) < \deg(q)$

$$p(A) = h(A) \cdot \chi_A(A) + r(A)$$

$h(A) \cdot \chi_A(A) = 0$

$r(A) \to \deg(r(A)) < n$

$$p(t) = \sum_{i=0}^{100} c_i \, t^i$$

$$p(t) = h(t) \cdot \chi_A(t) + r(t)$$

$$p(A) = r(A)$$

Wrong proof

$$\det(A - tI) = \chi_A(t)$$

$$\chi_A(A) = 0 \qquad \det(A - A \cdot I) = 0$$

$$\chi_A = \chi_T \quad , \quad U A U^* = T$$

$\chi_T(t) = 0$ → verify
→ induction

For any polynomial $p \in \mathbb{C}[z]$: $p(UAU^*) = U(p(A))U^*$

$$0 = \chi_T(T) = U(\chi_T(A))U^*$$

$$= U(\chi_A(A))U^*$$

$$0 = U\chi_A(A) \cdot U^* \Rightarrow \chi_A(A) = 0$$