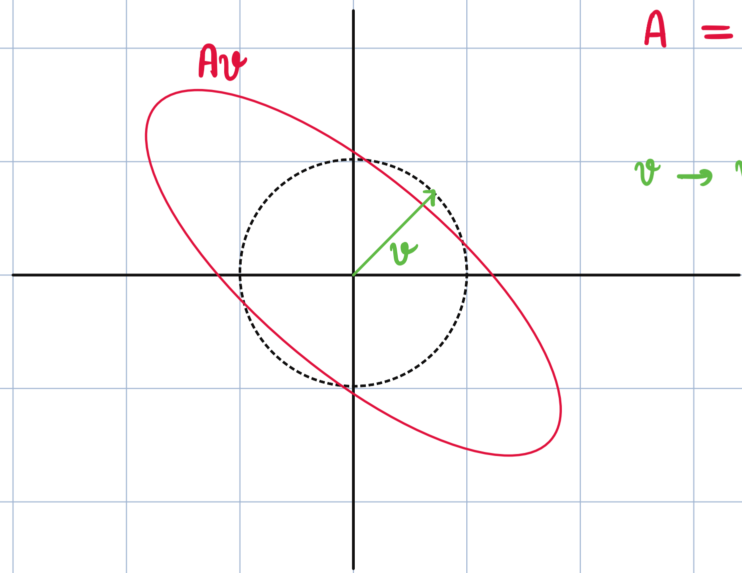
To - do

$\to \det(x) = \det(x^T)$

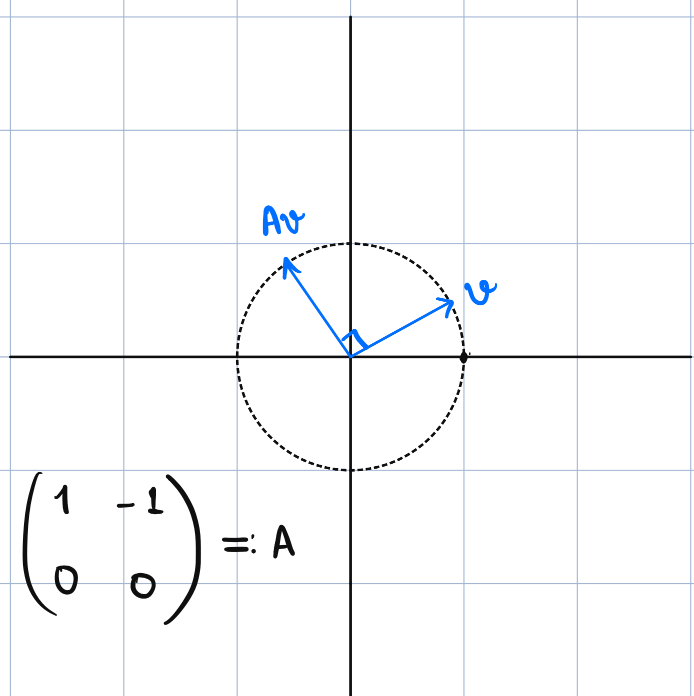
2024/11/05 - Linear Algebra - Week 14

$\to$ Geogebra



A = some matrix

$v \to$ vector of norm 1



$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} =: A$$

Simplest V.S $\to$ $\mathbb{R}$ $\{1 - 0\}$

LT $(\mathbb{R}) \rightsquigarrow$ scalar multiplication

atomic operation in LT $\to$ scalar multiplication

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \sim A$$

"nice" matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix}$$

$\to$ Spectral analysis of a matrix

light $\equiv$ matrix

break

7 colours

break into components

Eigenvalues and Eigenvectors

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $\lambda \in \mathbb{F}$ is said to be an eigenvalue of A if there exists a non-zero $v \in M_{n\times 1}(\mathbb{F})$ such that

existential def^n

$$A \cdot v = \lambda \cdot v$$

then such a vector is called an eigenvector for A with respect to the eigenvalue λ.

$\equiv$ LT $\rightsquigarrow$ $T \in L(V)$, $\lambda \in F$ s.t $T(v) = \lambda v$

$\rightarrow$ By def^n, v is non-zero, λ can be zero

Eigenvalues for linear transformation

Let V be a vector space over $\mathbb{C}$ and $T \in L(V)$. Then $\lambda \in \mathbb{C}$ is called an eigenvalue of T if there exists a non-zero vector $v \in V$ such that

$$T(v) = \lambda \cdot v$$

then such a vector is called an eigenvector for T with respect to the eigenvalue λ.

If λ is an eigenvalue of A then the space $\mathrm{Null}(A - \lambda I)$ is called the eigen-space of A w.r.t eigenvalue λ.

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underset{\substack{\downarrow \\ 1}}{\lambda} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

1 is an eigenvalue and any non-zero vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigen vector for A.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda I \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Non-trivial $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ only if $\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda I\right) = 0$

if det $\neq 0$

$\downarrow$

$v = 0$ is the only sol^n

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not allowed as an eigenvector

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)^2 - 4$$

$$(1-t)^2 - 4 = 0$$

$$\Rightarrow \quad 1 - t = \pm 2$$

$$\Rightarrow \quad t = 1 \pm 2$$

$$t = 3, -1$$

If $t = 3$

$$A - 3I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}}_{\text{rank} = 1}$$

nullity $= 1$

$\Downarrow$

non trivial solution for

$$\text{Solution} = \left\{ \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$

$t = -1$

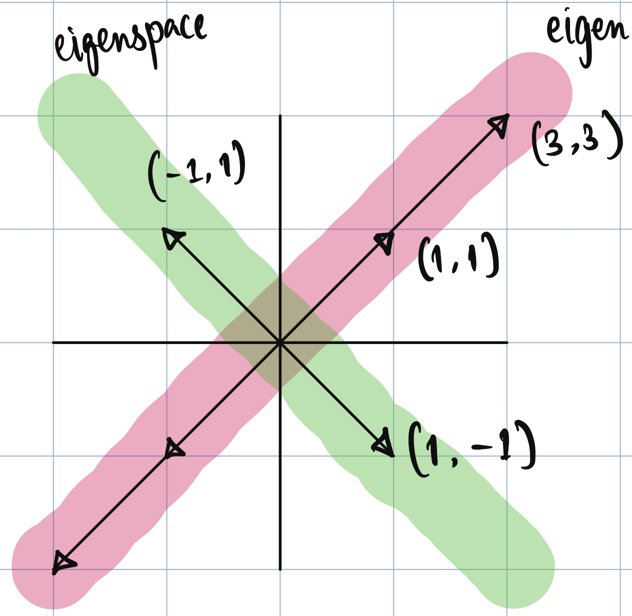
$$A - (-1)I$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}_{\downarrow}$$

rank $= 1$

$$\text{Solution} = \left\{ \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$



eigen space
↓
eigenvectors for a particular λ
$\cup$
0

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = 3 \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$$

Let $A \in M_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the <u>characteristic polynomial</u> of A is $\chi_A(t)$ given by

$$\chi_A(t) = \det(A - tI)$$

The roots of $\chi_A(t)$ are called <u>characteristic roots</u>.

$\rightarrow \boxed{\text{degree}\left(\chi_A(t)\right) = n}$

Exercise
Hint: inductively using co-factor minor expansion

Base case: $n = 1 \qquad A - tI = [a_{11} - t]$

$$\det(A - tI) = a_{11} - t$$

$$\text{degree} = 1 = n$$

Inductive hypothesis:

Let $\text{degree}\left(\chi_B(t)\right) = k$ where $B \in M_{k \times k}(\mathbb{F})$.

Let $A \in M_{(k+1)\times(k+1)}(\mathbb{F})$.

Then $\det(A - \lambda I) = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1k} & a_{1k+1} \\ \vdots & a_{22} - \lambda & & & & \vdots \\ \vdots & & a_{33} - \lambda & & & \vdots \\ a_{(k+1)1} & a_{(k+2)2} & \cdots & & & a_{(k+1)(k+1)} \end{bmatrix}$

$$= (a_{11} - \lambda) \cdot M_{11} - a_{12} \cdot M_{12} + \cdots + (-1)^{k+1} M_{1k} + (-1)^{k+2} M_{1\,k+1}$$

The terms $-a_{12} \cdot M_{12}, \ldots, (-1)^{k+2} M_{1\,k+1}$ have a degree atmost $(k-1)$.

$$\therefore \quad \text{degree}(\det(A - \lambda I)) = 1 + \deg(M_{11})$$

$$M_{11} = \det\left(\underbrace{\begin{bmatrix} a_{22} - \lambda & & \\ & \ddots & \\ & & a_{k+1\,k+1} - \lambda \end{bmatrix}}_{k \times k \text{ matrix}} \right)$$

$\deg(M_{11}) = k$ (Inductive hypothesis).

By fundamental theorem of algebra
$\chi_A(t)$ has atmost n real roots
and exactly n complex roots (counting multiplicity)

Proposition:
λ is an eigenvalue for A if and only if λ is a root of $\chi_A(t)$.

Proof: $(A - \lambda I) \cdot X = 0$ has a non-zero solution if and only if
$\det(A - \lambda I) = 0$ has real roots for λ.

$$\det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\chi_A(t) = \det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} -t & -1 \\ 1 & -t \end{bmatrix} \right)$$

$$= t^2 + 1 \rightsquigarrow$$ A does not have eigenvalue in $\mathbb{R}$

Corollary

$A \in M_n(\mathbb{C})$, then A has at least one eigenvalue.

Theorem: Let $A \in M_n(\mathbb{F})$. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be distinct eigenvalues of A. If v_i is an eigenvector w.r.t λ_i, $i = 1, \ldots, r$, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

Proof:

λ_1, λ_2, $\lambda_1 \neq \lambda_2$

$v_1 \quad v_2$

$\{v_1, v_2\}$ is linearly independent

If not

$$\alpha \cdot v_1 = v_2 \qquad \alpha \neq 0$$

so

$$A(\alpha \cdot v_1) = A(v_2)$$

$$\alpha \lambda_1 v_1 = \lambda_2 v_2$$

$$\Rightarrow \lambda_1 (\alpha v_1) = \lambda_2 v_2$$

$$\Rightarrow \boxed{\lambda_1 v_2 = \lambda_2 v_2}$$

$$\Rightarrow (\lambda_1 - \lambda_2) v_2 = 0$$

$\because$ $v_2 \neq 0$ by definition and $\lambda_1 \neq \lambda_2$, this gives a contradiction.

Hence $\{v_1, v_2\}$ is LI

Assume, $\{v_1, \dots, v_{k-1}\}$ is LI

$k-1 < r$

Claim: $v_k \notin \operatorname{span}\{v_1, \dots, v_{k-1}\}$

$\underbrace{\text{Principle of mathematical induction}}_{\text{true for all } n \in \mathbb{N}} \neq \underbrace{\text{inductive proof}}_{n \text{ is fixed}}$

If not: $\alpha_1, \dots, \alpha_{k-1}$ $\overbrace{\text{not all zero}}^{v_k \text{ eigenvectors}}$ such that

$$v_k = \sum_{i=1}^{k-1} \alpha_i v_i$$

$$A(v_k) = A\left(\sum_{i=1}^{k-1} \alpha_i v_i\right)$$

$$\lambda_k v_k = \sum_{i=1}^{k-1} \alpha_i \lambda_i v_i$$

$$\lambda_k \sum_{i=1}^{k-1} \alpha_i v_i = \sum_{i=1}^{k-1} \alpha_i \lambda_i v_i$$

$\lambda_k v_k$

$= \sum \alpha_i \lambda_i v_i$

$$\Rightarrow \quad \sum_{i=1}^{k-1} (\lambda_k - \lambda_i)\, \alpha_i v_i = 0$$

Since $\{v_1, \ldots, v_{k-1}\}$ is linearly independent

$$\underbrace{(\lambda_k - \lambda_i)}_{\substack{\text{non-zero} \\ \text{(distinct)}}}\, \alpha_i = 0 \qquad \forall\ i = 1, \ldots, k-1$$

at least one non-zero

$i_0 \rightarrow \alpha_{i_0} \neq 0$

$$\underbrace{(\lambda_k - \lambda_{i_0})}_{\text{non-zero}}\ \underbrace{\alpha_{i_0}}_{\text{non-zero}} = 0$$

$$\Rightarrow\Leftarrow$$

$\therefore \quad v_k \notin \operatorname{span}\{v_1, \ldots, v_k\}$

Hence, inductively $\{v_1, \dots, v_r\}$ is linearly independent.

Corollary: Sum of eigenspaces is a direct sum

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be distinct eigenvalues of $A \in M_n(\mathbb{F})$ $[\mathbb{F} = \mathbb{R}$ or $\mathbb{C}]$. Then the sum of space

$$\mathrm{Null}(A - \lambda_1 I) + \mathrm{Null}(A - \lambda_2 I) + \cdots + \mathrm{Null}(A - \lambda_r I)$$

is a direct sum.

Equivalently, if $v_i \in \mathrm{Null}(A - \lambda_i I)$, $i = 1, 2, \dots, r$, then the set $\{\lambda_1, \dots, \lambda_r\}$ is linearly independent.

① Let $Q \in M_n(\mathbb{R})$. Q is an orthogonal matrix if $Q^TQ = QQ^T = I$

If Q is orthogonal and λ is an eigenvalue for Q

then $|\lambda| = 1$.

Suppose $\exists$ $v \neq 0$ such that $Q(v) = \lambda v$

$$(Qv)^T(Qv) = (\lambda v)^T(\lambda v) = \lambda^2 v^T v$$

$$(Qv)^T(Qv) = v^T Q^T Q v = v^T v \qquad [\because Q \text{ is orthogonal}]$$

$$\therefore \quad \lambda^2 v^T v = v^T v$$

$$\Rightarrow \quad \lambda^2 = 1 \quad (\because v \neq 0) \Rightarrow |\lambda| = 1$$

② Let λ be an eigenvalue of $A \in M_n(\mathbb{F})$, then λ is also an eigenvalue for A^T.

Note: for any $X \in M_n(\mathbb{F})$,

$$\det(X) = \det(X^T)$$

For $\lambda \in \mathbb{F}$, $\det(A - \lambda I) = 0$

$$\Leftrightarrow \det((A - \lambda I)^T) = 0$$

$$\Leftrightarrow \det(A^T - \lambda I) = 0$$

Hence, λ is a root of $\chi_A(t)$ iff λ is a root of $\chi_{A^T}(t)$.

Equivalently, λ is an eigenvalue for A iff λ is an eigenvalue for A^T.

③ Given any $A \in M_n(\mathbb{C})$, if λ is an eigenvalue for A, then $\bar{\lambda}$ is an eigenvalue for A^*.

For $\lambda \in \mathbb{C}$, $\det(A - \lambda I) = 0$

$$\iff \overline{\det(A - \lambda I)} = 0$$

$$\iff \det((A - \lambda I)^*) = 0 \qquad \left[\because \overline{\det(P)} = \det(P^*) \ \forall\, P \in M_n(\mathbb{C})\right]$$

$$\iff \det(A^* - \bar{\lambda} I) = 0$$

Hence, λ is an eigenvalue of A iff $\bar{\lambda}$ is an eigenvalue for A^*

④ A matrix $A \in M_n(\mathbb{C})$ is said to be unitary if $U^* U = U U^* = I$.

Let λ be an eigenvalue for a unitary matrix U, then $|\lambda| = 1$

$\left(\Leftrightarrow \lambda = e^{i\theta}\right)$ $\theta \in [0, 2\pi)$

Proof: Similar to ①

08 Nov 2024

$\chi_A(t) = \det(A - tI) \rightsquigarrow$ analysis

$\chi_A(t) = \det(tI - A) \rightsquigarrow$ algebra

orthogonal matrices $Q^T Q = I \longrightarrow |\lambda| = 1$

unitary matrices $Q^* Q = I \longrightarrow |\lambda| = 1$

Let λ be an eigenvalue of $A \in M_n(\mathbb{F})$. Then, λ is also an eigenvalue of A^T

Exercise : $\det(A) = \det(A^T)$
$A \in M_n(\mathbb{C})$ $\quad \det(A^*) = \overline{\det(A)}$

λ is a root of characteristic for A

$\Leftrightarrow$ λ is a root of characteristic for A^T

λ is a root of $\chi_A(t) \iff \bar{\lambda}$ is a root of $\chi_{A^*}(t)$.

$A \in M_n(\mathbb{C})$

Suppose λ is an eigenvalue for A.
then $\bar{\lambda}$ is an eigenvalue for A^*.

Since λ is an eigenvalue of A

$$\det(A - \lambda I) = 0 \quad \text{—— } (1)$$

$$\begin{aligned} \det(A^* - \bar{\lambda} I) &= \det((A - \lambda I)^*) \\ &= \overline{\det(A - \lambda I)} \\ &= 0 \qquad (\because (1)) \end{aligned}$$

$\blacksquare$

Hermitian $A \in M_n(\mathbb{C})$ and $A = A^* = \overline{A^T}$

then, if λ is an eigenvalue for A, then $\lambda \in \mathbb{R}$

or $\lambda = \bar{\lambda}$

Since λ is an eigenvalue, $\exists$ non-zero $v \in M_{n \times 1}(\mathbb{C})$ such that

$$Av = v$$

$$v^* A v = v^*(Av) = v^* \lambda v = \lambda v^* v$$

$$v^* A v = v^* A^* v = (Av)^* v = \bar{\lambda} v^* v$$

($\because$ A hermitian)

$$\therefore \quad \lambda v^* v = \bar{\lambda} v^* v$$

$\because \quad v \neq 0, \quad v^* v \neq 0.$

$\therefore \quad \lambda = \bar{\lambda}$

or $\quad \lambda \in \mathbb{R}$ $\blacksquare$

Positive matrix ⟶ like unitary, orthogonal, hermition

Google page rank algorithm

- Hermition : $A = A^*$
- $\forall X \in M_{n\times 1}(\mathbb{F})$: $X^* A X \geq 0$

$$v^* A v \geq 0$$

$$\lambda \underbrace{v^* v}_{\text{inner product}} \geq 0$$

inner product ⟶ always ≥ 0

$$\lambda \geq 0$$

If all eigenvalues are +ve, then is the matrix positive?

No.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1-t & 1 \\ 0 & 1-t \end{bmatrix} \rightarrow (1-t)^2 = 0$$

$$\boxed{\lambda = 1}$$

Idempotent Matrix — Oblique Projection

$W_1 + W_2 = V \longleftrightarrow$

such that $W_1 + W_2$ is a direct sum

$v = w_1 + w_2$, here w_1 and w_2 are unique for v

A matrix $P \in M_n(\mathbb{C})$ is called an oblique projection if $P^2 = P$

$P : M_{n\times 1}(\mathbb{C}) \longrightarrow M_{n\times 1}(\mathbb{C})$

$V = M_{n\times 1}(\mathbb{C})$; assume $P^2 = P$

$W_1 = \mathrm{col}(P)$ $\quad W_2 = \mathrm{null}(P) = \mathrm{col}(I - P)$

P - oblique

$P_1(v) := w_1$

$P_2(v) := w_2$

$\boxed{\text{Exercise: } P_1, P_2 \text{ are linear maps}}$

P_i is called oblique projection of V onto W_i, $i = 1, 2$

$$P^2(v) = P(P(v)) = P(w_1)$$
$$= w_1 \quad (w_1 \text{ is its own representative in } W_1 + W_2)$$
$$= P(v)$$

$\boxed{P^2 = P}$ $\rightsquigarrow$ can be taken as a def^n

range$(P) = W$

$w \in W_1$

$P(w) = w$

Exercise: Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and W_1, W_2 be its two subspaces such that $W_1 \oplus W_2 = V$.

Then $\forall\, v \in V \; (= W_1 \oplus W_2)$, there exist unique $w_1 \in W_1$, $w_2 \in W_2$ such that $v = w_1 + w_2 \longrightarrow (1)$

Let $P_1 : V \longrightarrow W_1$ be a map defined by

$$P_1(v) = w_1 \,, \quad \text{where } v = w_1 + w_2 \,, \quad w_i \in W_i$$

We need to show that P_1 is a linear transformation on V.

$$\begin{aligned} P_1(p + \alpha q) &= P_1(w_{1p} + w_{2p} + \alpha w_{1q} + \alpha w_{2q}) \\ &= w_{1p} + \alpha w_{1q} \\ &= P_1(p) + \alpha P_1(q) \end{aligned}$$

Exercise: $P^2 = P$

then $(I - P)^2 = I - P$

$$(I-P)(I-P) = I(I-P) - P(I-P)$$
$$= I - P - \cancel{P} + \cancel{P}$$

So if P is an oblique projection then $I - P$ is also an oblique projection.

$W \subseteq V$, (V is an inner product space over $\mathbb{C}$).

$$W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0, \ \forall w \in W \}$$

Exercise: $W \subseteq V$

$$(W^{\perp})^{\perp} = W$$

$$W^{\perp} = Y = \{x \in V : \langle x, w \rangle = 0 \quad \forall \; w \in W\}$$

$$Y^{\perp} = \{v \in V : \langle v, y \rangle = 0 \;\wedge\; \langle y, w \rangle = 0 \quad \forall$$

$$\forall \; x \in (W^{\perp})^{\perp}$$

$$\langle x, y \rangle = 0 \quad \forall \quad y \in W^{\perp}$$

$$\Rightarrow \langle y, w \rangle = 0 \quad \forall \; w \in W$$

$$\langle x, y \rangle = 0 \quad \forall \; y$$

$W + W^{\perp} \longrightarrow$ is it direct sum?

Let $\quad w \in W \cap W^{\perp}$

$\langle w, w \rangle = 0$

$\Rightarrow \quad w = 0$

Hence $\quad W + W^{\perp}$ is a direct sum

$V \overset{?}{=} W + W^{\perp}$ $\qquad\qquad v$

$V \supseteq W + W^{\perp}$

To prove $\quad V \subseteq W + W^{\perp}$

$\boxed{\text{if } \dim(W + W^{\perp}) = \dim(V)}$ → works for finite dimensional space

exercise

<u>Claim:</u> $(W + W^{\perp})^{\perp} = \{0\}$

Suppose not, $\exists$ non-zero $v \in V$ s.t.

$$\langle v, (w_1 + w_2) \rangle = 0 \quad \forall \; w_1 \in W$$
and $w_2 \in W^{\perp}$

In particular $\langle v, w_1 \rangle = 0 \quad \forall \; w_1 \in W_1$

$\Rightarrow \quad v \in W^{\perp}$

and $\langle v, w_2 \rangle = 0 \quad \forall \; w_2 \in W_2$

$\Rightarrow \quad v \in W$

$\Rightarrow \quad v \in W \cap W^{\perp}$

$\Rightarrow \quad v = 0$

orthogonal direct sum

$*$ $\boxed{\boxed{W \oplus W^{\perp}} = V$ for all subspaces W of $V}$

$v = w_1 + w_2$, $\quad w_1 \in W_1$

$w_2 \in W^{\perp}$

$P(v) := w_1$

$P^2 = P$ for any direct sum (W_1, W_2)

$P = P^*$ (P is Hermitian for orthogonal projection)

Claim: P is Hermitian ; $P = P^*$

$T \in M_n(\mathbb{C})$

$\langle Tv, w \rangle$

$= w^* T v$

$\langle v, T^* w \rangle$

$= w^* (T^*)^* v$

$= w^* T v$

Claim: $P^* = P$

$v, w \in V$

$\langle Pv, w \rangle = \langle v, Pw \rangle$

If $P \in M_n(\mathbb{C})$

$P^* = \overline{P^T}$

Let $T \in L(V)$

then $T^* \in L(V)$

is defined to be the unique linear map S satisfying

$\langle T(v), w \rangle = \langle v, Sw \rangle$

$\langle Tv, w \rangle = \langle v, T^* w \rangle$

$w^* T v$

$\rightarrow$ eigenvalues of $P \rightarrow 0$ or 1

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Oblique Projection

Given: ① $W_1, W_2 \subseteq V \cong M_{n\times 1}(\mathbb{C})$

② $W_1 + W_2$ is a direct sum

③ $W_1 + W_2 = V$

Define: $P_1(v) = P_1(w_1 + w_2) = w_1$

$v = w_1 + w_2$ is a unique representation

$P_2(v) = P_2(w_1 + w_2) = w_2$

P_1 and P_2 are linear $\rightsquigarrow$ prove

$P_i^2 = P_i \quad , \; i = 1, 2$

$\text{col}(P_1) = W_1$

$\text{null}(P_1) = W_2 = \text{col}(P_2)$

Given: $W \subseteq V \cong M_{n\times 1}(\mathbb{C})$

Orthogonal Projections

$W^{\perp}$ exists

and $W + W^{\perp}$ is a direct sum $\rightarrow$ $\circledast$

and $V = W + W^{\perp}$

$$P_1(v) = P_1(w_1 + w_2) = w_1 \quad ; \quad w_1, w_2 \in W_1, W_2$$

$$P_2(v) = P_2(w_1 + w_2) = w_2$$

$P_1^2 = P_1$	$P_1^* = P_1$
$P_2^2 = P_2$	$P_2^* = P_2$

Let $P^2 = P$ and λ be an eigenvalue of P. Then $\exists$ a non-zero $v \in V$ st.

$$P(v) = \lambda v$$

$$P^2(v) = P(\lambda v)$$

$$= \lambda^2 v$$

$$P(v) = P^2(v)$$

$$\Rightarrow \quad \lambda v = \lambda^2 v$$

$$\Rightarrow \quad \lambda = \lambda^2$$

$$\hookrightarrow \boxed{\lambda = 0, 1}$$

$P_1(v) = w_1$ where $v = w_1 + w_2$, $w_1 \in W$, $w_2 \in W^{\perp}$

and $v = w_1 + w_2$ is unique representation (from ⊛)

$P_1 = P_1^*$ (Matrix representation of P_1) $\rightsquigarrow$ $(P_1 - P_1^*) = 0$

Result: $A \in M_{n\times n}(\mathbb{C})$ and for every $v, w \in M_{n\times 1}(\mathbb{C})$

$$w^* A v = 0 \implies A = 0$$

$$w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}_{1\times n} \begin{bmatrix} a_{11} & a_{12} & & \\ & \ddots & & \\ & & & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}_{n\times 1}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}_{n\times 1}$$

$$= a_{11}$$

$a_{mn} = e_m^* A e_n$; e_m is m^{th} canonical basis

For arbitrary $v, w \in M_{n\times 1}(\mathbb{C})$

$$w^* (P_1 - P_1^*) v$$

$$= w^* P_1 v - w^* P_1^* v \longrightarrow ⊛$$

now, $w = w_1 + w_2$, $w_1 \in W$ $w_2 \in W^{\perp}$

$v = v_1 + v_2$, $v_1 \in W$ $v_2 \in W^{\perp}$

So $w^* P_1 v = (w_1 + w_2)^* P_1 (v_1 + v_2)$

$= (w_1^* + w_2^*) P_1 (v_1 + v_2)$

$= w_1^* P_1 (v_1 + v_2) + w_2^* P_1 (v_1 + v_2)$

$= w_1^* v_1 + w_2^* v_1$

$w_2 \in W^{\perp}$, $v_1 \in W$. Hence $\langle v_1, w_2 \rangle = w_2^* v_1 = 0$

$w^* P_1 v = w_1^* v_1 \longrightarrow (1)$

|||ly $\quad w^* P_1^* v = (P_1(w))^* v = w_1^* v$

$$= w_1^* (v_1 + v_2)$$

$$= w_1^* v_1 \longrightarrow (2)$$

From (*), (1) and (2)

$$w^* (P_1 - P_1^*) v = 0 \quad \forall \; w, v \in M_{n\times 1}(\mathbb{C}).$$

$$\therefore \quad P_1 = P_1^*$$

$$\therefore \quad \boxed{P_i = P_i^2 \quad \text{and} \quad P_i = P_i^*}$$

$$P_2 = I - P_1$$

Def^n: A matrix $P \in M_n(\mathbb{C})$ is called an orthogonal projection

$$P^2 = P$$

$$P^* = P$$

and $W = \text{col}(P)$

Why eigenvalues $\rightsquigarrow$ to get some linearly independent vectors

$\lambda_1, \dots, \lambda_r$

$\downarrow \qquad \downarrow$

$v_1, \dots, v_r \rightarrow$ orthogonal

Block matrix $\longrightarrow$ Triangularization $\longrightarrow$ Diagonalization

① Symm.

② Hermitian

Block matrices $\longrightarrow$ reverse of partition of matrices

$$A : V_1 \oplus V_2 \longrightarrow W_1 \oplus W_2$$

Here $\oplus$ represents a orthogonal sum

$$A = \begin{array}{cc} \begin{array}{cc} V_1 & \oplus \; V_2 \end{array} & \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & \begin{array}{c} W_1 \\ \oplus \\ W_2 \end{array} \end{array}$$

$A_{11}(v_1) = w_1 \rightsquigarrow$ orthogonal projection w.r.t W_1, $W_2 = W^{\perp}$

$V = V_1 \oplus V_2$

$v = v_1 + v_2$

vectors

$A \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$A = \begin{array}{cc} V_1 & V_2 \end{array}$$
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{array}{c} W_1 \\ W_2 \end{array}$$

$A(v) =$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{12} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} v_1 + A_{12} v_2 \\ A_{21} v_1 + A_{21} v_2 \end{bmatrix} \in W$$

$$V = W \oplus W^{\perp}$$

$$A = \begin{array}{c|c} \overset{W \;\oplus\; W^{\perp}}{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}} & \begin{matrix} W \\ \oplus \\ W^{\perp} \end{matrix} \end{array}$$

$$W = \{ \alpha \cdot e_1 \mid \alpha \in \mathbb{C} \}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \beta \\ \beta \end{pmatrix}$$

If $A \in M_n(\mathbb{C})$, $W \subseteq M_{n\times 1}(\mathbb{C})$,

$$A(W) \subseteq W$$

then $$A^*(W^\perp) \subseteq W^\perp$$

<u>Schur Triangularization</u>

$$Q \in M_n(\mathbb{C})$$

$$Q^* Q = I = QQ^*$$

$*$ $$Q = [q_1 \mid q_2 \mid \dots \mid q_n], \quad q_i \in M_{n\times 1}(\mathbb{C})$$

$\{q_1, \dots, q_n\}$ is an orthonormal basis for $M_{n\times 1}(\mathbb{C})$.

⊛ $A \in M_n(\mathbb{C})$, $Q^* A Q = \tilde{A}$

$$\begin{aligned}
\chi_{\tilde{A}}(t) &= \det(Q^* A Q - tI) \\
&= \det(Q^* AQ - t\,Q^* Q) \\
&= \det(Q^* (A - tI) Q) \\
&= \det(Q^*) \cdot \det(A - tI) \cdot \det(Q)
\end{aligned}$$

$\because$ $Q^* Q = I$, $\det(Q^*) \cdot \det(Q) = 1$

$\therefore$ $\chi_{\tilde{A}}(t) = \det(A - tI) = \chi_A(t)$

let U be an invertible matrix $(\in M_n(\mathbb{C}))$, then for any $A \in M_n(\mathbb{C})$,

$$\chi_{U^{-1}AU}(t) = \chi_A(t)$$

$U = [u_1 | \cdots | u_n]$, $\{u_1, \ldots, u_n\}$ is a basis for $M_{n\times 1}(\mathbb{C})$.

---

let $U^{-1} A U = \tilde{A}$

then $AU = U\tilde{A}$

$\Rightarrow A[u_1 | u_2 | \cdots | u_n] = U\tilde{A}$

$\Rightarrow [A(u_1) | A(u_2) | \cdots | A(u_n)] = U\tilde{A}$

Suppose $\tilde{A}$ is diagonal if $A_{ij} \neq 0$ only if $i = j$

$$\left[A u_1 \mid A u_2 \mid \cdots \mid A u_n \right] = \left[u_1 \mid \cdots \mid u_n \right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= \left[\lambda_1 u_1 \mid \lambda_2 u_2 \mid \cdots \mid \lambda_n u_n \right]$$

$$A(u_i) = \lambda_i u_i$$

$$\boxed{\begin{array}{c} Q^* A Q = \tilde{A} \\ \Downarrow \\ A(q_i) = \lambda_i q_i \end{array}} \longrightarrow \text{distinct } \lambda_i \text{ s} \Rightarrow \text{matrix diagonalizable}$$

Ex : $\quad \mathrm{tr}(AB) = \mathrm{tr}(BA)$