2024/10/29 - Linear Algebra - Week 13

<u>Recall</u>: Let $v, w \in M_{n\times 1}(\mathbb{C})$

$$\langle v, w \rangle = w^* v$$

$$= \begin{bmatrix} \overline{w_1} & \overline{w_2} & \overline{w_3} & \cdots & \overline{w_n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\|v\|^2 = \sum_{i=1}^{n} |v_i|^2$$

Let $\{e_1, \dots, e_n\}$ be the canonical basis $M_{n\times 1}(\mathbb{C})$

then,

$$\langle e_i, e_j \rangle := e_j^* e_i = \delta_{ij}$$ (Kronecker's delta) defined for $M_{n\times 1}(\mathbb{C})$

$\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$

Orthonormal basis

let $\mathcal{B} \subseteq V$ be a basis, then $\mathcal{B} = \{f_1, f_2, \dots f_n\}$ is said to be orthonormal basis of V if

$$\langle f_i, f_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

If $\langle f_i, f_j \rangle = 0 \ \forall \ i \neq j$, then $\mathcal{B}$ is called orthogonal basis.

$$\underbrace{\langle e_i, e_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}_{\text{orthonormal}}$$

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e.g.: For $\mathbb{R}^2$, let $\mathcal{B} = \{(1, 1), (-1, 1)\}$ be a basis

then $\langle (1, 1), (-1, 1) \rangle = 0$

but $\langle (1, 1), (1, 1) \rangle \neq 1$

norm = 1

$\therefore$ orthogonal but not ortho<u>normal</u>

$$\text{let} \quad \mathcal{B} = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$

normalization

$$\|(1,1)\| = \sqrt{2}$$

$$\frac{(1,1)}{\|(1,1)\|} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

orthogonal basis $\longrightarrow$ divide by norm $\longrightarrow$ orthonormal basis

In general, if $\langle v, w \rangle = 0$ then $\langle \lambda \cdot v, \mu \cdot w \rangle = 0 \quad \forall \lambda, \mu \in \mathbb{C}$

$$\mathcal{B} = \{ b_1, \dots, b_n \} \subseteq M_{n \times 1}(\mathbb{C})$$

If $v \in M_{n \times 1}(\mathbb{C})$

$$v = \sum_{i=1}^{n} \alpha_i b_i$$

$$\alpha_1 \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$\downarrow$

System of eq^n

$$[b_{ij}] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Let V be an inner product space, then any two subspaces $W_1, W_2 \subseteq V$ are said to be mutually orthogonal if for any $x \in W_1$ and $y \in W_2$, $\langle x, y \rangle = 0$

Observation: In an inner product space, knowing an orthonormal basis helps in easy identification of coordinates.

Let V be an inner product space over $\mathbb{C}$, and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be an orthonormal basis for V. Let v be an ordinary vector in V. Since $\mathcal{B}$ is a basis, there exist unique $v_1, v_2, \ldots, v_n \in \mathbb{C}$ such that

$$v = \sum_{i=1}^{n} v_i b_i$$

But since $\mathcal{B}$ is orthonormal, $\langle b_i, b_j \rangle = \delta_{ij}$ for $j = 1, \dots, n$

$$\langle v, b_j \rangle = \left\langle \sum_{i=1}^{n} v_i b_i \, , \, b_j \right\rangle$$

$$= \sum_{i=1}^{n} v_i \langle b_i , b_j \rangle$$

$$= v_j$$

So, the coordinate representation of any $v \in V$ with respect to an orthonormal basis $\mathcal{B} = \{ b_1, \dots, b_n \}$ is given by

$$v \equiv \left(\langle v, b_1 \rangle, \langle v, b_2 \rangle , \dots , \langle v, b_n \rangle \right)$$

If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is given orthonormal basis

$$v = \sum_{i=1}^{n} \alpha_i b_i$$

Let $j = 1, 2, \ldots, n$

$$\langle v, b_j \rangle = \left\langle \sum_{i=1}^{n} \alpha_i b_i \right\rangle = \sum_{i=1}^{n} \alpha_i \langle b_i, b_j \rangle = \alpha_j \langle b_j, b_j \rangle$$

$$= \alpha_j$$

$$\therefore \quad \boxed{v = \sum_{i=1}^{n} \langle v, b_i \rangle b_i}$$

If $\mathcal{B}$ is only orthogonal:

$$v = \sum_{i=1}^{n} \frac{\langle v, b_i \rangle}{\langle b_i, b_i \rangle} \cdot b_i$$

* How to identify an orthonormal basis?
* How to construct an orthonormal basis?
  ↳ G.S.

Proposition

Let V be an inner product space over $\mathbb{C}$ (or $\mathbb{R}$). Then, any set $S = \{b_1, b_2, \dots, b_n\} \subseteq V$ of non-zero mutually orthogonal vectors are linearly independent.

Maths

→ linear in 1st coordinate

→ conjugate linearity

Physics

→ linear in 2nd coordinate

$$v = \sum_{i=1}^{n} \frac{\langle b_i, v \rangle}{\langle b_i, b_i \rangle} b_i$$

mutually orthogonal
↓
pairwise orthogonal

Proof: Let $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ be mutually orthogonal non-zero vectors.

Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that $\sum_{i=1}^{n} \alpha_i b_i = 0$

Fix j in between $1, \dots, n$. Then

$$\left\langle \sum_{i=1}^{n} \alpha_i b_i , b_j \right\rangle = 0 \qquad \left\{ \because \langle 0, v\rangle = \langle v, 0\rangle = 0 \right.$$

$$\Rightarrow \quad \alpha_j \langle b_j , b_j \rangle = 0 \qquad \left\{ \because \langle b_i, b_j\rangle = 0 \; i \neq j \right.$$

$$\Rightarrow \quad \alpha_j = 0 \qquad \left\{ \because \langle b_j, b_j\rangle \neq 0 \text{ as } b_j \neq 0 \right.$$

So, each $\alpha_1, \dots, \alpha_n$ must be zero. $\{b_1, \dots, b_n\}$ are linearly independent. If V is n-dimensional, $\mathcal{B} \rightarrow$ basis

e.g.: dimension 5

$\{b_1, b_2, \dots b_5\}$

$\downarrow$

non-zero orthogonal vectors

$\Downarrow$

basis

$w_1, \dots, w_n \in W_1, \dots, W_n$

$\downarrow$

orthogonal

$\Rightarrow$ sum is direct

Corollary: Let $W_1, W_2, \ldots, W_n \subseteq V$ be mutually orthogonal subspaces of the inner product space V (non-zero) over $\mathbb{C}$. Then the sum of spaces $W_1 + W_2 + \cdots + W_n$ is a direct sum.

Proof: From the following equivalence (discussed earlier)

$W_1 + W_2 + \cdots + W_n$ is direct if and only if for any non-zero $v_i \in W_i$, $i = 1, 2, \ldots, n$; the set $\{v_1, \ldots, v_n\}$ is is linearly independent.

Note: any n mutually orthogonal vectors form a basis of an n-dim. V.S.

IMP Warning: Let $v_1, v_2, v_3 \in V$. Then, $\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = 0$ does not mean $\langle v_1, v_3 \rangle = 0$

Gram - Schmidt Process

Let V be an inner product space over $\mathbb{C}$ and $\mathcal{B} = \{b_1, \dots, b_n\}$ be any basis of V. The following vectors defined inductively form an orthogonal basis:

$$u_1 = b_1 \quad \text{and} \quad u_k = b_k - \underbrace{\left(\sum_{j=1}^{k-1} \frac{\langle b_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right)}_{\text{partial coordinate expression w.r.t } \{u_1, \dots, u_{k-1}\}}, \quad n \geq k > 1$$

Let $\mathcal{B}' = \{u_1, u_2, \dots, u_n\}$. Then $\mathcal{B}'$ is an orthogonal basis for V.

$$u_1 = b_1$$

$$u_2 = b_2 - \left(\frac{\langle b_2, u_1 \rangle}{\langle u_1, u_2 \rangle} u_1 \right) = b_2 - \frac{\langle b_2, b_1 \rangle}{\|b_1\|^2} b_1$$

$$u_3 = b_3 - \left(\frac{\langle b_3, u_2 \rangle}{\|u_2\|^2} u_2 + \frac{\langle b_3, u_1 \rangle}{\|b_1\|^2} b_1 \right)$$

$$\vdots$$

$$u_k = b_k - \left(\sum_{j=1}^{k-1} \frac{\langle b_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right)$$

$$\vdots$$

$$u_n = b_n - \left(\sum_{j=1}^{n} \frac{\langle b_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right)$$

Claim: $\{u_1, \ldots, u_n\} \longrightarrow$ an orthogonal basis

Show

① $u_j \neq 0 \quad \forall\ j = 1, \ldots, n$

i.e. u_j cannot be zero

② $\langle u_i, u_j \rangle = 0 \quad$ if $\quad i \neq j$

basis element

$$u_1 = b_1 \longrightarrow \text{non-zero}$$

$$u_2 = b_2 - \left(\frac{\langle b_2, u_1 \rangle}{\langle b_2, u_1 \rangle} b_1 \right)$$

Suppose $u_2 = 0$

$$b_2 = \frac{\langle b_2, u_1 \rangle}{\langle b_2, u_1 \rangle} b_1$$

$\Rightarrow \Leftarrow$

Basis LI

$u_1 \neq 0$

$u_2 \neq 0$

Suppose $u_3 = 0$

$$b_3 = \frac{\langle b_3, u_1 \rangle}{\langle b_3, u_1 \rangle} b_1 + \frac{\langle b_3, u_2 \rangle}{\langle b_3, u_2 \rangle} u_2$$

$$\frac{\langle b_3, u_2 \rangle}{\langle b_3, u_2 \rangle} \left(b_2 - \frac{\langle b_2, u_1 \rangle}{\langle b_2, u_1 \rangle} b_1 \right)$$

$$\text{span}\{u_1, u_2\} = \text{span}\{b_1, b_2\}$$

If $b_3 \in \text{span}\{u_1, u_2\}$
$\Longrightarrow b_3 \in \text{span}\{b_1, b_2\}$ $\longrightarrow$ contradicts linear independence of B

Show that $u_j \neq 0$ for any $j = 1, 2, \dots, n$

Mutually orthogonal $\longrightarrow$ Program

Claim : $\langle u_2, u_1 \rangle = 0$

$$\left\langle b_2 - \frac{\langle b_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 \, , \, u_1 \right\rangle$$

$$\langle b_2 , u_1 \rangle - \frac{\langle b_2 , u_1 \rangle}{\cancel{\langle u_1, u_1 \rangle}} \cancel{\langle u_1, u_1 \rangle}$$

$$= 0$$

$u_3 \perp u_2$ and $u_3 \perp u_1 \longrightarrow$ not transitive

$$u_2 \perp u_1 \not\Rightarrow u_3 \perp u_1$$
$$u_3 \perp u_2$$

$\langle u_3, u_2 \rangle$

$$= \left\langle b_3 - \frac{\langle b_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle b_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 \, , \, u_2 \right\rangle$$

$$= \langle b_3, u_2 \rangle - \frac{\langle b_3, u_2 \rangle}{\cancel{\langle u_2, u_2 \rangle}} \cancel{\langle u_2, u_2 \rangle} - \frac{\langle b_3, u_1 \rangle}{\langle u_1, u_1 \rangle} \cancelto{0}{\langle u_1, u_2 \rangle}$$

$$= 0$$

Show that $u_k \perp u_1, u_2, \dots, u_{k-1}$

Projection method

QR Factorization of Full Rank Matrix

Let $A \in M_{n\times n}(\mathbb{C})$ be a full rank matrix. Then there exists matrix $Q, R \in M_{n\times n}(\mathbb{C})$ such that $Q^* Q = I$, R is upper triangular and $A = QR$

Let $A \in M_{n\times n}(\mathbb{C})$ have full rank. Equivalently, if $A = [A_1 | A_2 | \cdots | A_n]$ with each $A_i \in M_{n\times 1}(\mathbb{F})$; $i = 1, \dots, n$ then $\{A_1, \dots, A_n\}$ is a basis for $M_{n\times 1}(\mathbb{C})$.

Using Gram Schmidt: $u_1 = A_1$

$$u_k = A_k - \left(\sum_{j=1}^{k} \frac{\overbrace{\langle A_k, u_j \rangle}^{u_j^* A_k}}{\langle u_j, u_j \rangle} u_j \right), \quad k = 2, \dots, n$$

is an orthogonal basis.

Equivalently,

$$A_1 = u_1$$

$$A_k = u_k + \sum_{j=1}^{k-1} \frac{u_j^* A_k}{u_j^* u_j} u_j$$

Then, for $u = [u_1 \mid u_2 \mid \dots \mid u_n]_{n \times n}$ and $G = \begin{bmatrix} 1 & \frac{u_1^* A_2}{u_1^* u_1} & \frac{u_1^* A_3}{u_1^* u_1} & \cdots & \frac{u_1^* A_n}{u_1^* u_1} \\ 0 & 1 & \frac{u_2^* A_3}{u_2^* u_2} & \cdots & \vdots \\ \vdots & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

we have, $u \cdot G = A$

Further, define $\quad Q_k = \dfrac{u_k}{(u_k^* \cdot u_k)^{1/2}} \qquad k = 1, \dots, n$

then $\quad Q_1, Q_2, \dots, Q_n \quad$ are orthonormal.

Then, the matrix $\quad Q := [\, Q_1 \mid Q_2 \mid \cdots \mid Q_n \,] \in M_{n\times n}(\mathbb{C})$

and $\quad Q^* Q = \begin{bmatrix} Q_1^* \\ \hline Q_2^* \\ \hline \vdots \\ \hline Q_n^* \end{bmatrix} [\, Q_1 \mid Q_2 \mid \cdots \mid Q_n \,]$

$$= I$$

Also
$$A_k = u_k + \left(\sum_{i=1}^{k-1} \frac{u_i^* A_k}{u_i^* u_i} u_i \right)$$

$$= Q_k \cdot \left(u_k^* \cdot u_k \right) + \sum_{i=1}^{k-1} Q_i \left(u_i^* A_k \right)$$

o To-do