2024/10/25 — Linear Algebra — Week 12

Inner Product Space

An inner product for a vector space V over $\mathbb{R}$ or $\mathbb{C}$ is defined to be a map $(v, w) \longmapsto \langle v, w \rangle$ where $v, w \in V$ and $\langle x, y \rangle \in \mathbb{R}$ (or $\mathbb{C}$) such that

(i) $\langle v, v \rangle \geq 0$, $\forall\ v \in V$ and $\langle v, v \rangle = 0 \iff v = 0$

(ii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$, $v, w \in V$

(iii) $\langle v_1 + \alpha v_2, w \rangle = \langle v_1, w \rangle + \alpha \langle v_2, w \rangle$, $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), $v, w \in V$

The vector space V with the inner product $\langle\ ,\ \rangle$ is called an inner product space.

<u>**Observation**</u>: Since $\langle v, w \rangle = \overline{\langle w, v \rangle}$, we notice that

$$\langle v, w_1 + \alpha w_2 \rangle = \overline{\langle w_1 + \alpha_2 w_2 , v \rangle}$$

$$= \overline{\langle w_1 , v \rangle} + \overline{\alpha_2} \, \overline{\langle w_2 , v \rangle}$$

$$= \langle v, w_1 \rangle + \overline{\alpha_2} \langle v, w_2 \rangle \quad , \quad v, w_1, w_2 \in V$$

and $\langle 0 , v \rangle = \langle v, 0 \rangle = 0 \quad , \quad \forall \ v \in V$

Example: Euclidean Space

Let $\mathbb{C}^n$ be the co-ordinate space over $\mathbb{C}$. We define the Euclidean inner product as

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

$$\overline{\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle} = \sum_{i=1}^{n} \overline{x_i} y_i$$

$$= \langle (y_1, \dots, y_n), (x_1, \dots, x_n) \rangle$$

* Verify that the above definition is indeed an inner product

The Euclidean space $\mathbb{R}^n$ over $\mathbb{R}$ has the inner product

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i$$

<u>Remark</u>: Every vector space inherits this inner product via a coordinate representation. (What about the cannonicity ?) and hence we will mostly restrict ourselves to $\mathbb{C}^n$ and $\mathbb{R}^n$

$\boxed{\text{IMP}}$ Euclidean inner product on $M_{n\times 1}(\mathbb{R})$ is

columns

$$\langle X, Y \rangle = Y^T X \qquad \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Euclidean inner product on $M_{n\times 1}(\mathbb{C})$ is

$$\langle X, Y\rangle = Y^* X \longrightarrow \begin{bmatrix} \overline{y_1} & \overline{y_2} & \cdots & \overline{y_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where $Y^* = \overline{Y^T}$; bar denotes complex conjugate done entrywise.

Proposition

Let V be an inner product space over $\mathbb{R}$ (or $\mathbb{C}$). Then for any vectors $v_1, \ldots, v_m$, $w_1, \ldots, w_n$ of V

$$\left\langle \sum_{i=1}^{m} \alpha_i v_i , \sum_{j=1}^{n} \beta_j w_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \overline{\beta_j} \langle v_i , w_j \rangle$$

Proof: follows directly from definition

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→ Euclidean space for complex coordinates

→ $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle \neq \langle (y_1, \dots, y_n), (x_1, \dots, x_n) \rangle$

→ conjugate symmetry

→ linearity in first coordinate + conjugate symmetry $\Longrightarrow$ linearity in both coordinates

$$\left\langle \sum_{i=1}^{n} \alpha_i v_i , \sum_{j=1}^{n} \beta_j v_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \overline{\beta_i} \, v_i w_j$$

$$\sum_{i=1}^{n} \alpha_i \left\langle v_i , \sum_{j=1}^{n} \beta_j w_j \right\rangle = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} \overline{\beta_j} \langle v_i , w_j \rangle \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i , w_j \rangle$$

inner product ~ angle b/w vectors

length ~ norm

Norm:

A norm on a vector space over $\mathbb{R}$ or $\mathbb{C}$ is defined to be a function $v \longmapsto \|v\|$ from V to $\mathbb{R}$ such that

(i) $\|v\| \geq 0 \quad \forall \ v \in V$ and $\|v\| = 0 \iff v = 0$ — positive definite

(ii) $\|\alpha \cdot v\| = |\alpha| \cdot \|v\|$ — homogeneity

(iii) $\|v + w\| \leq \|v\| + \|w\|, \quad v, w \in V$ — triangle inequality

Observation: Follows from the defⁿ:

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x - y\| \leq \|x\| + \|y\|$$

$v \xmapsto{\|\cdot\|} \|v\|$

→ non-negative real number

→ $\|\cdot\| = 0$

$\iff \cdot = 0$

→ norm changes proportionally

$\|\alpha \cdot v\| = |\alpha|\,\|v\|$

v is scaled by scalar α, $\|v\|$ is scaled by $|\alpha|$

$$\|x + y\| \leq \|x\| + \|y\|$$

$$\big| \|x\| - \|y\| \big| \leq \|x - y\| \leq \|x\| + \|y\| \rightarrow \text{exercise}$$

Here
Euclidean norm
mostly

$\|x - y\|$

Examples of norm on $\mathbb{C}^n$ (and $\mathbb{R}^n$)

1

One norm

$$\|(x_1, \dots, x_n)\|_1 = \sum_{i=1}^{n} |x_i|$$

3 properties: homogeneity, positivity, triangle inequality followed ✓

This is called the 1-norm

e.g.: $\underbrace{\|(1, 1)\|_1}_{\in \mathbb{C}^2} = 2$

② ∞ norm

$$\|(x_1, \dots, x_n)\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

③ p-norm $p=2$

$$\|(x_1, \dots, x_n)\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \longleftrightarrow \text{Euclidean norm}$$

$p=2$ is called the Euclidean norm

$$\mathbb{R}^2 \; ; \quad \|(x_1, x_2)\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$$

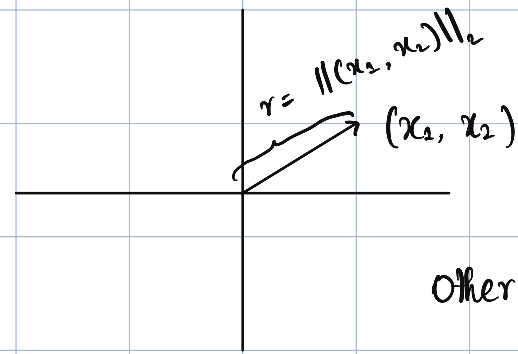
$$\|(x_1, x_2)\|_2 = r \;, \quad r \in \mathbb{R}^+ \cup \{0\}$$

If $r = 0$, then $(x_1, x_2) = (0, 0)$

$r \neq 0$, $\quad \|(x_1, x_2)\|^2 = |x_1|^2 + |x_2|^2 = r^2$

or $x_1^2 + x_2^2 = r$ $\longrightarrow$ Circle with radius r and centre O

$\{ (x_1, x_2) \mid \|(x_1, x_2)\|_2 = r \}$ is a circle of radius r centered at 0.



$(x_1, x_2) \in \mathbb{C}^2$

$$\|(x_1, x_2)\|_2 := \left(|x_1|^2 + |x_2|^2 \right)^{1/2}$$

$$\Rightarrow \quad \|(x_1, x_2)\|_2^2 = |x_1|^2 + |x_2|^2$$

$$= x_1 \overline{x_1} + x_2 \overline{x_2}$$

$$= \langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle$$

Other norms
$\downarrow$
different shapes

$\xleftarrow{\text{parallelogram law}}$

inner product exists $\Rightarrow$ norm can always be defined

useful in <u>optimizations</u>

limited resource $\}$ inner product + norm

* Not every norm is induced by an inner product

- A norm induced by an inner product must satisfy a property called the <u>parallelogram law</u>:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$\downarrow$$

$$\langle x+y,\ x+y\rangle^2 + \langle x-y,\ x-y\rangle^2$$

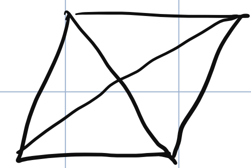
$$\big(\langle x,\ x+y\rangle + \langle y,\ x+y\rangle\big)^2$$

$$\Big(\overbrace{\langle x, x\rangle}^{a} + \overbrace{\langle x, y\rangle + \langle y, x\rangle}^{b} + \langle y, y\rangle\Big)^2$$

$$+ \big(\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle\big)^2$$

$$= (a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$$

$$= 2\langle x, x\rangle^2 + 2\langle y, y\rangle^2$$



Parallelogram

Σ length of diagonals

$= \Sigma$ sides