2024/10/15 - Linear Algebra - Week 11 ✓

Corollary: If A has full column rank, then $\text{rank}(AB) = \text{rank}(B)$

Proof: If A has full column rank, then A is left invertible

$\exists U$ s.t $\underset{n\times m}{U}\underset{m\times n}{A} = I$

Let U be some left inverse of A.

So $B = (U \cdot A)\, B = U \cdot (A\, B)$

row rank (AB) = row rank (B)

right / left inverses constructed from spanning sets

→ no unique inverse always (unique if invertible)

$\boxed{\text{Q7 of midsem}}$

$$U(A \cdot B) = B$$

$$\text{row}(U \cdot A \cdot B) \subseteq \text{row}(AB)$$

$$\Rightarrow \text{row}(B) \subseteq \text{row}(AB) \subseteq \text{row}(B)$$

$$\Rightarrow \text{row}(B) = \text{row}(AB)$$

$$\therefore \text{row rank}(B) = \text{row rank}(AB)$$

$$\text{rank}(B) = \text{rank}(AB)$$

$\text{row}(PQ) \subseteq \text{row}(Q)$

row rank $\equiv$ rank

column space version

$AB \rightarrow$ right invertible

$$\text{column rank}(A) = \text{column rank}(AB)$$

Remark: Applying the above result to transpose of matrices, of the above result we show that if B has full row rank then $\text{rank}(AB) = \text{rank}(A)$

Proposition: Let $A \in M_{m \times n}(\mathbb{F})$ and $P \in M_m(\mathbb{F})$ and $Q \in M_{n \times n}(\mathbb{F})$ be invertible matrices, then

$$\text{rank}(PAQ) = \text{rank}(A)$$

Rank is invariant under multiplication by invertible matrices

Nullity of a matrix $\longrightarrow$ dimension of null space

Let $\quad \text{Null}(A) = \{X \in M_{n\times 1}(\mathbb{F}) : A_{m\times n} X_{n\times 1} = 0\}$

then $\dim(\text{Null}(A))$ is called the nullity of A, denoted by nullity(A).

$$A_{m\times n} X_{n\times 1} = 0$$

$\nu = \text{nullity}(A) \leq n \qquad\qquad = n \Rightarrow A = 0$

If $A \neq 0$, $\quad \nu < n$

$$\text{Null}(A) \subset M_{n\times 1}(\mathbb{F})$$

$\boxed{\text{Rank - Nullity theorem}}$

Let $A \in M_{m\times n}(\mathbb{F})$, then $\text{rank}(A) = n - \text{Nullity}(A)$

or Let V, W be a vector space over $\mathbb{F}$ and let $T \in L(V, W)$, then

$$\dim(T(V)) = \dim(V) - \dim(\ker(T))$$

Proof: $A \in M_{m\times n}(\mathbb{F})$. Let $\dim(\mathrm{Null}(A)) = \nu \geqslant 1$

Let $\{b_1, \dots, b_\nu\}$ be a basis for $\mathrm{Null}(A) \subseteq M_{n\times 1}(\mathbb{F})$

So there exists a basis $\mathcal{B}$ of $M_{n\times 1}(\mathbb{F})$ with $\{b_1, \dots, b_\nu\} \subseteq \mathcal{B}$

Let $\mathcal{B} = \{b_1, \dots, b_\nu, b_{\nu+1}, \dots, b_n\}$ be a basis of $M_{n\times 1}(\mathbb{F})$

Claim: $\{A(b_\nu), A(b_{\nu+1}), \dots, A(b_n)\}$ is a basis for $\mathrm{col}(A)$

AX

$$X = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$$

$$AX = \underbrace{\alpha_1 A(b_1) + \alpha_2 A(b_2) + \dots + \alpha_\nu A(b_\nu)}_{=\,0} + \alpha_{\nu+1} A(b_{\nu+1}) + \dots + \alpha_n A(b_n)$$

Spanning set

$$z \in col(A)$$

$$z = A \cdot X$$

$$X = \sum_{i=1}^{n} \alpha_i b_i \quad ; \quad \mathcal{B} = \{ \underbrace{b_1, \dots, b_\nu}_{\text{basis for } Null(A)}, \dots, b_n \}$$

$$AX = A \left(\sum_{i=1}^{n} \alpha_i b_i \right)$$

$$= \sum_{i=1}^{n} \alpha_i A b_i$$

$$= \sum_{i=1}^{n-\nu} \alpha_i A b_{\nu+i}$$

linearly independent

$$\boxed{\sum_{i=1}^{n-r} \alpha_i \, A(b_{r+i}) = 0} \rightarrow \text{①}$$

To prove

$$\Longrightarrow \quad \alpha_1 = 0 = \cdots = \alpha_n$$

$$\sum_{i=1}^{n-r} \alpha_i \, A(b_{r+i}) = A\left(\sum_{i=1}^{n-r} \alpha_i b_{r+i}\right) \longrightarrow \text{⑪}$$

$$\therefore \quad \sum_{i=1}^{n-r} \alpha_i \, b_{r+i} \in \mathrm{Null}(A)$$

$\because$ $\boxed{\mathcal{B} \text{ is a basis}}$, each $\alpha_i = 0$

$\therefore$ $\{A(b_{r+1}), \ldots, A(b_n)\}$ is a basis for $\mathrm{col}(A)$

$$\therefore \quad |\mathcal{B}| = n = \nu + n - \nu$$

$$= \nu + \dim(\operatorname{col}(A))$$

$$n = \text{Nullity}(A) + \text{rank}(A)$$

<u>Consequences</u>

* Let $A \in M_{m\times n}(\mathbb{F})$ with $n > m$, then

$$\text{Nullity}(A) > 0$$

$$\text{rank}(A) \leq m < n$$

Hence, by rank - nullity theorem,

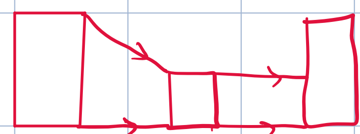
$$\text{nullity}(A) > 0$$

$A_{3\times 4}$

$\text{rank}(A) \leq 3$

$\underbrace{\text{rank}(A)}_{\leq 3} + \underbrace{\text{nullity}(A)}_{\geq 1} = 4$

So if $A \cdot X = 0$ denotes a system of equations with number of variables strictly more than the number of equations, then non trivial solutions must exist.



* let $T \in L(V, W)$, if $\dim(W) > \dim(V)$ then T can never be surjective

$[T]_{\mathcal{B}_2 \times \mathcal{B}_1}$ $\qquad$ $\operatorname{rank}\left([T]_{\mathcal{B}_2 \times \mathcal{B}_1}\right) \leqslant |\mathcal{B}_1| < |\mathcal{B}_2|$

$$\boxed{\begin{array}{l} T(x) = 0 \\ \not\Rightarrow \quad x = 0 \end{array}}$$

* Let $A \in M_{n \times n}(\mathbb{F})$, then the following are equivalent

(i) $\text{rank}(A) = n$

(ii) A is injective

(iii) A is surjective

(iv) A is invertible

Proof: (i) $\to$ (ii) $\to$ (iii) $\to$ (iv)

Rank factorization (CR - factorization) → 1^{st} factor full column rank → left invertible
column
row
→ 2^{nd} factor full row rank → right invertible

Theorem: Let $A \in M_{m \times n}(\mathbb{F})$ and if $\text{rank}(A) = r \geq 1$ then there exist $P \in M_{m \times r}(\mathbb{F})$ and $Q \in M_{r \times n}(\mathbb{F})$ such that

(a) $\text{rank}(P) = \text{rank}(Q) = r$ and

(b) $A = PQ$

Remark: Rank factorization is not unique, the choice depends on bases. But once P is fixed, Q is also fixed. The pair (P, Q) is unique.

$A_{m \times n}$ has rank (m)

$A_{m \times n} U_{n \times m} = I$

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use : $A \rightarrow$ models some system

$A_{1024 \times 2056} \longrightarrow 1024 \times 2056$ parameters

$\downarrow$

rank $-$ 100

$A = P_{1024 \times 100} \cdot Q_{100 \times 2056}$

Proof: Given $\underbrace{\text{rank}(A)}_{\text{dim. of row space} = \text{dim of col space}} = r$

dim of col space $\rightarrow |\text{basis}(\text{col space})| = r$

Choose a basis $\mathcal{B} = \{P_1, P_2, \ldots, P_r\}$ of col(A) $\subseteq M_{m \times 1}(\mathbb{F})$

$A_{m \times n}$ rank r

$\downarrow$

$\exists$ some $P \in M_{m \times r}(\mathbb{F})$ and $Q \in M_{r \times n}(\mathbb{F})$

such that

$\text{rank}(P) = \text{rank}(Q) = r$

and

$A = PQ$

$A_{3 \times 4} \rightarrow \text{rank}(A) = 2$

$P_{3 \times 2} \; Q_{2 \times 4} = A$

$\downarrow \text{rank}(P) = \text{rank}(Q) = 2$

$\leftarrow j^{th}$ column

Then, A_j can be written as

$$A_j = \sum_{i=1}^{r} q_{ij} P_i \qquad (\text{for } j = 1, 2, \dots, n)$$

Let $\quad Q_j = [\, q_{1j} \quad q_{2j} \quad \cdots \quad q_{rj} \,]^T \quad , \quad j = 1, \dots, n$

Construct the matrices $\quad P := [\, P_1 \mid P_2 \mid \cdots \mid P_r \,]_{m \times r}$ $\quad \longrightarrow$ chosen to be LI rank $\to r$

$$Q := [\, Q_1 \mid Q_2 \mid \cdots \mid Q_n \,]_{r \times n}$$

then $\qquad A = P \cdot Q.$

$$P_{m \times r} \qquad Q_{r \times n}$$

$$r = \text{rank}(PQ) \leq \underbrace{\min \{ \text{rank } P, \text{ rank } Q \}}_{\text{Q7} \,\to\, \text{col space of } PQ \text{ is contained in col space of } Q} \leq r$$

col space (P) = col space (A)

null space (A) = null space (Q)

full column rank
$\downarrow$
left invertible

$P \rightarrow$ left invertible

$Q \rightarrow$ right invertible

Let $x \in \text{Null}(A)$

$\Rightarrow \quad A \cdot x = 0$

$\Rightarrow \quad (PQ)\, x = 0$ P has full column rank $\longleftrightarrow$ P is left invertible

P is left invertible. $\exists\, P'$ s.t $P'P = I$

$\Rightarrow \quad (P'P)\, Q\, x = 0$

$\Rightarrow \quad Q \cdot x = 0 \Rightarrow x \in \text{Null}(Q)$

$\therefore \quad \text{Null}(A) \subseteq \text{Null}(Q) \quad \text{—} \; (1)$

Suppose $\quad x \in \text{Null}(Q)$

$\Rightarrow \quad Q(x) = 0$

$\Rightarrow \quad A(x) = P\,\underline{Q(x)} = 0$

$\text{Null}(Q) \subseteq \text{Null}(A) \longrightarrow (2)$

$(1) \;\&\; (2) \Rightarrow \boxed{\text{Null}(A) = \text{Null}(Q)}$

<u>OR</u>: Rank nullity $\longrightarrow$ P $\rightarrow$ rank r

$\rightarrow$ nullity 0 $\Rightarrow$ $P\,\underbrace{Q(x)}_{\text{has to be zero}} = 0$

How to find P and Q ?

$\rightarrow$ Find basis of column space

$\rightarrow$ ...

Note : P, Q need not be fixed,
once P is fixed,
Q is fixed ($Q \rightarrow$ coordinates in expression of $A = \cdots P_i$)

$\boxed{\text{IMPORTANT}}$

Let P — left invertible
Q — right invertible

$$\text{rank}\,(P \cdot A \cdot Q) = \text{rank}\,(A)$$

$$B = \overset{\text{investible}}{P \cdot A \cdot Q}$$

rank same

Example

Let $A \in M_{2\times 3}(\mathbb{R})$ be a matrix with

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

find matrices P and Q that provide the rank factorization of A.

$$2\times 2 \quad \times \quad \underline{2\times 3}$$

Let $P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Suppose $A_{m\times n}$, $B_{m\times n}$ and $\text{rank}(A) = \text{rank}(B)$. Then $\exists$ P, Q <u>invertible</u> s.t. $P \cdot A \cdot Q = B$

$P, Q \rightarrow$ transition matrix

$I : V_{\mathcal{B}} \longrightarrow V_{\mathcal{B}'}$

$$\left[\quad\right] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[I]_{\underset{n}{\mathcal{B}'} \times \underset{n}{\mathcal{B}}} = \overbrace{\left[\quad\right] \cdot}^{\text{does not 'change' anything}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

rank(n)

$= \text{first column of } [I]_{\mathcal{B}' \times \mathcal{B}}$

$\updownarrow$

coordinate of b_1 w.r.t $\mathcal{B}'$

Change of coordinates

Let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered bases of a vector space V of dimension n. Let $I \in L(V)$ be the identity map. The matrix $[I]_{\mathcal{B}' \times \mathcal{B}}$ is called the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Recall: $I : V_{\mathcal{B}} \longrightarrow V_{\mathcal{B}'}$

If $e_j = \sum_{i=1}^{n} \beta_{ij} e_i'$, then for $v = \sum_{i=1}^{n} v_i e_i$, we have

$$I(v) = \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{n} \beta_{ij} e_i' \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} v_j e_i'$$

So, if coordinates of v w.r.t $\mathcal{B}$ are $(v_1, \dots, v_n)$, then $\left(\sum_{j=1}^{n} \beta_{1j} v_j, \sum_{j=1}^{n} \beta_{2j} v_j, \dots, \sum_{j=1}^{n} \beta_{nj} v_j\right)$ is the coordinate of V w.r.t $\mathcal{B}'$.

The matrix $[I]_{\mathcal{B}' \times \mathcal{B}} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \beta_{21} & \ddots & \vdots \\ \vdots & & \\ \beta_{n1} & \cdots & \beta_{nn} \end{bmatrix}$ is called the the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Let $P \in M_n(\mathbb{F})$ be invertible, then P is a transition matrix for the cannonical basis of $\mathbb{F}^n$ to the basis of $\mathbb{F}^n$ given by the columns of P

$$\begin{tikzcd} & \mathcal{B}_1 \\ V \arrow[ur] \arrow[dr] & \\ & \mathcal{B}_1' \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{B}_2 \\ W \arrow[ur] \arrow[dr] & \\ & \mathcal{B}_2' \end{tikzcd}$$

$$V \xrightarrow{I} V \xrightarrow{T} W \xrightarrow{I} W$$

P A Q

$$[T]_{\mathcal{B}_2 \times \mathcal{B}_1} = [I]_{\mathcal{B}_2 \times \mathcal{B}_2'} [T]_{\mathcal{B}_2' \times \mathcal{B}_1'} [I]_{\mathcal{B}_1' \times \mathcal{B}_1}$$

$$T : V_{\mathcal{B}_1} \to V_{\mathcal{B}_2}$$

$$[T]_{\mathcal{B}_1' \times \mathcal{B}_1'} = [I]_{\mathcal{B}_1' \times \mathcal{B}_1} [T]_{\mathcal{B}_1 \times \mathcal{B}_1} [I]_{\mathcal{B}_1 \times \mathcal{B}_1'}$$

$$= [I]^{-1}_{\mathcal{B}_1 \times \mathcal{B}_1'} [T]_{\mathcal{B}_1 \times \mathcal{B}_2} [I]_{\mathcal{B}_1 \times \mathcal{B}_1'}$$

Equivalent matrices

Two matrices $A, B \in M_{m \times n}(\mathbb{F})$ are said to be equivalent if there exist invertible matrices such that $B = Q^{-1} A P$; $Q \in M_{m \times m}(\mathbb{F})$, $P \in M_{n \times n}(\mathbb{F})$.

Two matrices $A, B \in M_n(\mathbb{F})$ are said to be similar if there exists invertible matrix P such that $B = P^{-1} A P$

* If B is similar to A, then $\det(B) = \det(A)$

Theorem (Rank Invariance)

Let $A, B \in M_{m \times n}(\mathbb{F})$, then the following are equivalent

(a) $B = Q \cdot A \cdot P$ (P, Q invertible matrices of appropriate size)

(b) $\text{rank}(A) = \text{rank}(B)$

Proof: We have already shown (a) $\Rightarrow$ (b)

⊛ $A, B \in M_{m \times n}(\mathbb{F})$

Let $\text{rank}(A) = r = \text{rank}(B)$

By rank nullity theorem,

$$n = \text{rank}(A) + \text{nullity}(A)$$

$$= \text{rank}(B) + \text{nullity}(B)$$

$$\Longrightarrow \quad \text{nullity}(A) = \text{nullity}(B)$$

$\mathcal{B}_1' \leftarrow \text{Null}(A) \subseteq M_{n\times 1}(\mathbb{F}) \qquad \text{col}(A) \subseteq M_{m\times 1}(\mathbb{F})$

$\mathcal{B}_2' \leftarrow \text{Null}(B) \subseteq M_{n\times 1}(\mathbb{F}) \qquad \text{col}(B) \subseteq M_{m\times 1}(\mathbb{F})$

Let $\mathcal{B}_1$ be a basis of $M_{n\times 1}(\mathbb{F})$ containing a basis of Null(A).

Let $\mathcal{B}_2$ be a basis of $M_{n\times 1}(\mathbb{F})$ containing a basis of Null(B).

$\mathcal{B}_1' \subseteq \mathcal{B}_1$

$\mathcal{B}_2' \subseteq \mathcal{B}_2$

$|\mathcal{B}_1| = |\mathcal{B}_2| = n$

$|\mathcal{B}_1'| = |\mathcal{B}_2'|$

Let C_1 be a basis of $M_{m\times 1}(\mathbb{F})$ containing a basis C_1' of $\mathrm{col}(A)$

Let C_2 be a basis of $M_{m\times 1}(\mathbb{F})$ containing a basis C_2' of $\mathrm{col}(B)$

$|C_1| = |C_2| = m$

$|C_1'| = |C_2'|$

Let $\mathcal{B}_1 = \{ b_1', b_2', \dots, b_{n-r}', b_{n-r+1}, \dots, b_n \}$

$\mathcal{B}_2 =$

Proof to-do