2024/10/01 - Linear Algebra - Week 10

Problem Sheet:

Proposition: $W_1, \dots, W_n \subseteq V$ subspace then $W_1 + \dots + W_n$ is direct iff for any $x_i \in W_i \setminus \{0\}$ then $\{x_1, \dots, x_n\}$ is linearly independent.

<u>Def^n</u>: Let $W_1, \dots, W_n \subseteq V$ be subspaces, then $W_1 + \dots + W_n$ is said to be direct if for any $v \in W_1 + \dots + W_n$ there exist unique $x_i \in W_i$, $i = 1, \dots, n$ such that

$$v = \sum_{i=1}^{n} \alpha_i x_i \quad ; \quad \alpha_i \in \mathbb{F}$$

Corollary: Let V be a vector space and $W_1, W_2, \dots, W_n$ be subspaces of V_1, then the following are equivalent.

(a) $W_1 + W_2 + \dots + W_n$ is a <u>direct sum</u>.

(b) $(W_1 + W_2 + \dots + W_i) \cap W_{i+1} = \{0\}$, $1 \le i \le n-1$

(c) $x_1 + x_2 + \dots + x_n \in W_1 + W_2 + \dots + W_n = 0$

$\Rightarrow x_1 = 0 = x_2 = \dots = x_n$

(d) $\dim(W_1 + \dots + W_n) = \sum_{i=1}^{n} \dim(W_i)$

<u>Proof</u>: By induction

$\rightarrow$ See problem set 11.

Exercise: Let $W_1, W_2, W_3 \subseteq V$ be finite dimensional subspaces. Then find $\dim(W_1 + W_2 + W_3)$.

$\boxed{\text{To-do}}$

Note: $W_1, W_2, W_3 \subseteq V$ and $W_1 \cap W_2 = \{0\} = W_2 \cap W_3 = W_1 \cap W_3$ does not imply $W_1 + W_2 + W_3$ is direct.

Partition of matrices

Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 & 6 \\ 3 & 8 & 3 \\ 0 & 0 & 0 \\ 4 & 3 & -5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & -1 & 6 \\ 3 & 8 & 3 \end{bmatrix}$$

We can partition the matrices A, B and perform multiplication.

$$A = \left[\begin{array}{c|c|c} I_{2\times 2} & \begin{matrix} -3 \\ 4 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right] \qquad B = \left[\begin{array}{ccc} -4 & -1 & 6 \\ -3 & 8 & 3 \\ \hline 0 & 0 & 0 \\ \hline 4 & 3 & -5 \end{array}\right]$$

$$A \cdot B = I_{2\times 2} \begin{bmatrix} 4 & -1 & 6 \\ 3 & 8 & 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}_{2\times 1} \cdot 0_{1\times 3} + 0_{2\times 1} \cdot \begin{bmatrix} 4 & 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & 6 \\ 3 & 8 & 3 \end{bmatrix}$$

Let $T \in L(V, W)$ and suppose that there exist $V_1, V_2 \subseteq V$ and $W_1, W_2 \subseteq W$ such that $V_1 \oplus V_2 = V$ and $W_1 \oplus W_2 = W$.

Then w.r.t base $\mathcal{B}_1, \mathcal{B}_2$ of V_1, V_2 and $\mathcal{B}_1', \mathcal{B}_2'$ of W_1, W_2 we can partition the matrix representation of T.

<u>Recall</u>: Since $V_1 \oplus V_2 = V$, $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V and similarly, $\mathcal{B}_1' \cup \mathcal{B}_2'$ is a basis for W.

Let $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$, $\mathcal{B}' := \mathcal{B}_1' \cup \mathcal{B}_2'$, then T has the matrix representation:

$$[T]_{\mathcal{B}' \times \mathcal{B}} = \begin{array}{c c} & \begin{array}{cc} \mathcal{B}_1 & \mathcal{B}_2 \end{array} \\ \begin{array}{c} \mathcal{B}_1' \\ \mathcal{B}_2' \end{array} & \left[\begin{array}{c|c} [T_{11}] & [T_{12}] \\ \hline [T_{21}] & [T_{22}] \end{array}\right] \end{array}$$

$T_{11} : V_1 \to W_1$

$T_{12} : V_2 \to W_1$

$T_{21} : V_1 \to W_2$

$T_{22} : V_2 \to W_2$

* Multiplication and Division rules hold as usual

Rank

Let V be a vector space over $\mathbb{F}$. We begin by defining 2 notions of ranks and then show that they are equivalent.

Row rank: Let $A \in M_{m\times n}(\mathbb{F})$, then row rank of A is defined to be the dimension of the row space of A

Column rank: Let $A \in M_{m\times n}(\mathbb{F})$, then column rank of A is defined to be the dimension of the column space of A

Row space $\rightarrow$ $\text{row}(A) = \{ X\cdot A \mid X \in M_{1\times m}(\mathbb{F}) \}$

Column space $\rightarrow$ $\text{col}(A) = \{ A \cdot X \mid X \in M_{n\times 1}(\mathbb{F}) \}$

Observe that $\dim(\text{row}(A)) \leq m$ and $\dim(\text{col}(A)) \leq n$.

Proposition: [Row rank vs. Column rank]

Let $A \in M_{m\times n}(\mathbb{F})$, then $\dim(\text{row}(A)) = \dim(\text{col}(A))$

Proof: Suppose $\dim(\text{row}(A)) = r$

$\dim(\text{col}(A)) = s$

Let $\mathcal{B} = \{b_1, b_2, b_3, \ldots b_s\} \subseteq M_{n\times 1}(\mathbb{F})$ be an ordered basis of $\text{col}(A)$.

For the matrix U defined as:

$$U = [\, b_1 \mid b_2 \mid \cdots \mid b_s \,]_{m \times s},$$

we have $\quad Y_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{si} \end{bmatrix} \in M_{s\times 1}(\mathbb{F})$ such that

$$A_i = U Y_i$$

Fix the matrix $Y = [\, Y_1 \mid Y_2 \mid \cdots \mid Y_n \,]_{s\times n}$, then by the above equation we have,

$$A = U Y$$

$$\operatorname{col}(A) \subseteq \operatorname{col}(U)$$

[from exam Q7]

$$\text{and} \quad \operatorname{row}(A) \subseteq \operatorname{row}(Y)$$

so $$\dim(\operatorname{row}(A)) \leq \dim(\operatorname{row}(Y))$$

$$\Rightarrow \quad r \leq s \longrightarrow (1)$$

Now, we interchange the roles of rows and columns:

$$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_m \end{bmatrix}, \quad \text{each } A_j \in M_{1\times n}(\mathbb{F})$$

Since $\dim(\text{row}(A)) = r$, there exist $\mathcal{B}' = \{C_1, C_2, \dots, C_r\}$, $Y' \in M_{m\times r}(\mathbb{F})$, such that

$$V := \begin{bmatrix} C_1 \\ \hline C_2 \\ \vdots \\ C_r \end{bmatrix}$$

and $\quad A = Y'V$

Then,

$$\dim(\text{col}(A)) \leq \dim(\text{col}(Y'))$$

$$s \leq r \longrightarrow \text{(ii)}$$

So by (i) and (ii), we conclude that $r = s$.

How to find rank ✓

Proposition

Let $A \in M_{n \times n}(\mathbb{F})$, then $\text{rank}(A) = n$ if and only if $\det A \neq 0 \iff A$ is invertible if and only if $\text{rank}(A) = n$.

Proof: to-do.

Proposition: Let $A \in M_{p \times q}(\mathbb{F})$ and $B \in M_{q \times r}(\mathbb{F})$, then

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$

Proof: Recall from midsem exam that

$$\text{row}(AB) \subseteq \text{row}(B) \quad ??$$

$$\Rightarrow \quad \dim(\text{row}(AB)) \leq \dim(\text{row}(B))$$

$$\operatorname{col}(AB) \subseteq \operatorname{col}(A)$$

$$\Rightarrow \quad \dim(\operatorname{col}(AB)) \leq \dim(\operatorname{col}(A))$$

So $\quad \operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$

$\blacksquare$

Full row rank: Let $A \in M_{m \times n}(\mathbb{F})$, then A is said to be of full row rank if $\operatorname{rank}(A) = m$.

Full column rank: Let $A \in M_{m \times n}(\mathbb{F})$, then A is said to be of full column rank if $\operatorname{rank}(A) = n$.

Proposition: Let $A \in M_{m \times n}(\mathbb{F})$, then the following are equivalent:

(a) A has full row rank.

(b) A is right invertible, i.e., $XA = YA \Rightarrow X = Y$, for appropriate matrices X and Y.

(c) $\mathrm{col}(A)$ is $M_{m \times 1}(\mathbb{F})$.

Proof: $(a \Rightarrow b)$. Suppose A has full row rank, i.e., the row $A_1, \ldots, A_m$ of A are linearly independent in $M_{1 \times n}(\mathbb{F})$.

We claim there exists matrix $U \in M_{n \times m}(\mathbb{F})$ such that

$$A_{m \times n} U_{n \times m} = I_{m \times m} \quad \text{—} (*)$$

Note: $m \leq n$, hence there is a non-trivial solution to the following equations

$$A(x_i) = e_i \quad , \quad e_i \in M_{m\times 1}(\mathbb{F})$$

For the system to be consistent we need all rows to contain a pivot.

Set $U := [\, u_1 \mid u_2 \mid \cdots \mid u_m \,]_{n\times m}$, where $u_i \in M_{n\times 1}(\mathbb{F})$ satisfies the above equations, then

$$A \cdot U = I$$

($XA = XY \Rightarrow X = Y$ is evident now)

$(b \Rightarrow c)$ Let's assume that $\exists\ U \in M_{n\times m}(\mathbb{F})$ s.t. $AU = I_{m\times m}$

Then (c) is evident, as for the columns U_i of U

We have

$$\sum_{j=1}^{n} U_{ji}\,(A_j) = e_i \; ; \quad \text{where} \quad e_i \in M_{m\times 1}(\mathbb{F})$$

is the i^{th} vector of canonical ordered basis

so $\operatorname{span}(\{A_j \mid j = 1, \dots, n\}) = M_{m\times 1}(\mathbb{F})$

$(c \Rightarrow a)$ Suppose $\operatorname{span}(\{A_j \mid j = 1, \dots, n\}) = M_{m\times 1}(\mathbb{F})$

since colum rank is same as row rank, the claim holds

true

Note: solution to the above system is not unique. In particular, the right inverse is also not unique for rectangular matrices.