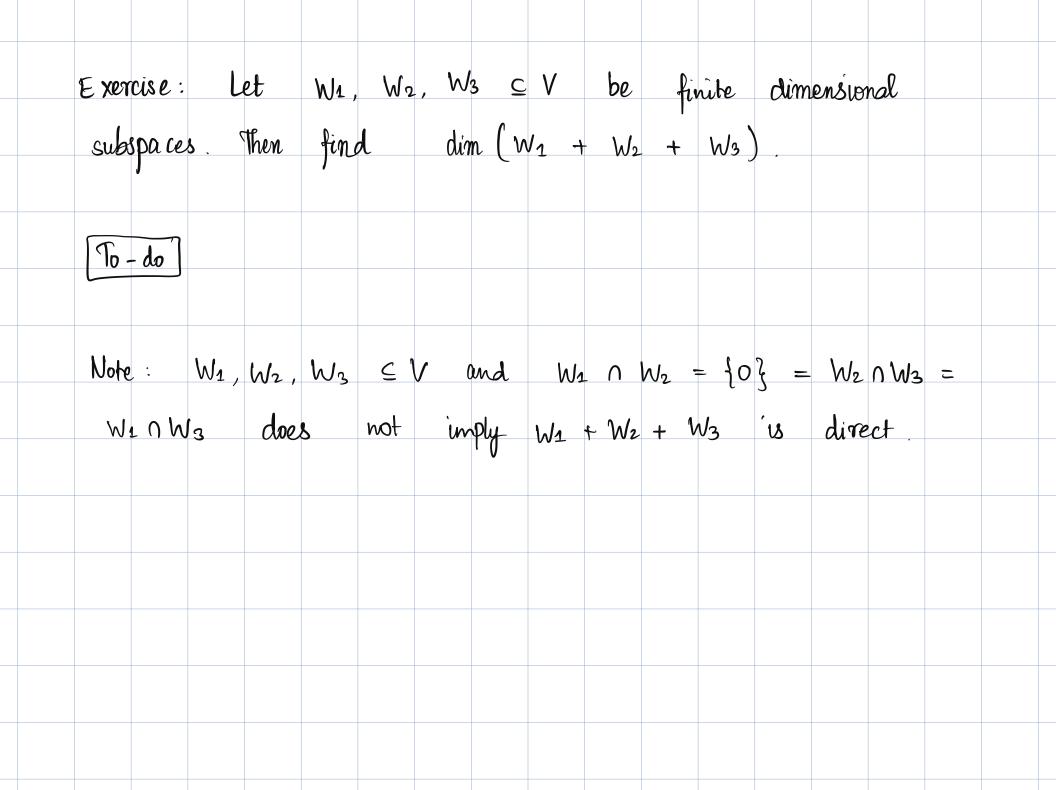
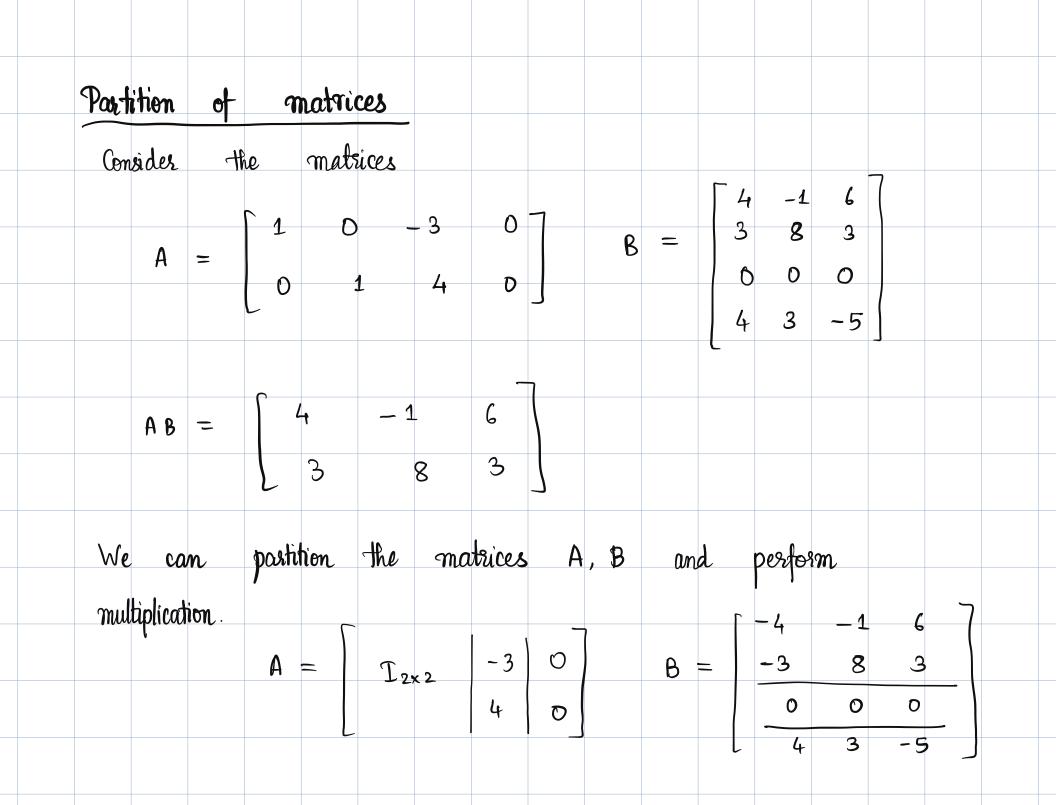
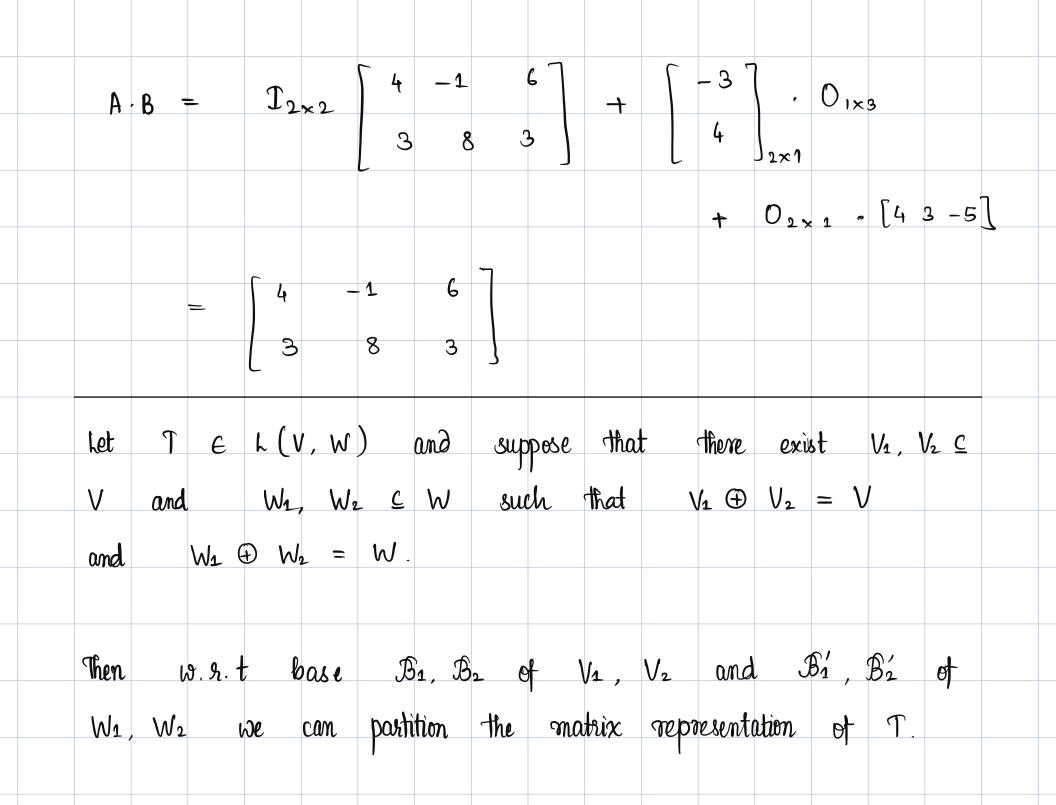
Problem					0 9				
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Defn: L	et W1	, , W	m <u>C</u> \	I be	subspaces	s, then	\mathcal{W}_1	+ + W,	1
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Corollary: Let V be a vector space and W1, W2, ..., Wn be subspaces of V1, then the following are equivalent. (a) W1 + W2 + ··· + Wn 'is a direct sum. (b) $(W_1 + W_2 + \cdots + W_i) \cap W_{i+1} = \mathcal{A} \mathcal{O} \mathcal{Y}$, $1 \leq i \leq n-1$ $(C) \quad \chi_2 + \chi_2 + \cdots + \chi_n \quad \in \quad W_1 + W_2 + \cdots + W_n = 0$ $\Rightarrow \chi_1 = 0 = \chi_2 = \dots = \chi_n$ (d) $\dim (W_1 + ... + W_n) = \sum \dim (W_i)$ Proof: By induction ---> See problem set 11.







Rank

Let V be a vector space over IF. We begin by defining 2

notions of ranks and then show that they are equivalent.

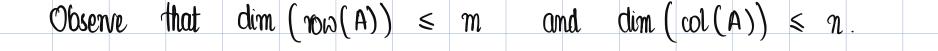
Row rank: Let A E Mmxn (IF), then sow rank of A is

defined to be the dimension of the row space of A

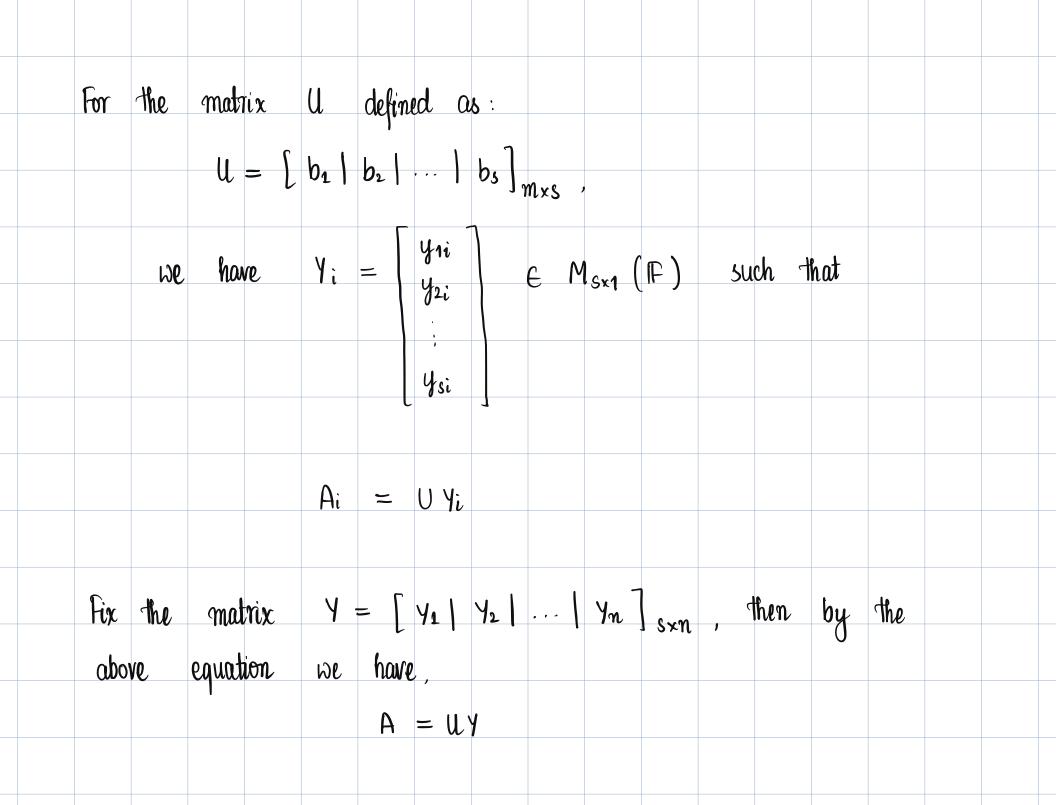
 $\frac{\text{Column rank}}{\text{defined to be the dimension of the column space of A}$

Row space $\rightarrow \operatorname{row}(A) = \{ \chi \cdot A \mid \chi \in M_{1\times m}(F) \}$

 $Column \quad \text{space} \rightarrow col(A) = \{A \cdot X \mid X \in M_{n \times 1}(F)\}$



Proposition: [Row rank vs. Column rank] het $A \in M_{m \times n}$ (F), then $\dim(row(A)) = \dim(col(A))$ **Proof**: Suppose $\dim(row(A)) = r$ $\dim (col(A)) = S$ Let $\mathcal{B} = \{b_1, b_2, b_3, \dots, b_s\} \subseteq M_{n \times 1}(\mathbb{F})$ be an ordered basis of col(A).



 $col(A) \subseteq col(U)$ [from exam] Q7 and row $(A) \subseteq row (Y)$ $\dim (row (A)) \leq \dim (row (Y))$ $\Rightarrow \qquad \gamma \leq S \qquad -$ SD (\mathbf{i}) Now, we interchange the roles of rows and columns: $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, each A_j \in M_{1 \times n} (\mathbb{P})$

Since dim (row (A)) = r, there exist
$$\mathcal{B}' = \{C_1, C_2, ..., C_r\},$$

 $Y' \in M_{mxr}(\mathbb{F}), \text{ such that}$
 $V := \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_r \end{bmatrix}$
and $A = Y'V$
Then, dim (col(A)) $\leq \dim (col(Y'))$
 $S \leq \gamma$ \longrightarrow (i)
So by (i) and (i), we conclude that $r = S$.

Propesition

 Let
$$A \in M_{n \times n} (\mathbb{F})$$
, then $\operatorname{vank} (A) = n$, if and only if

 det $A \neq O \iff A$ is invertible if and only if $\operatorname{vank} (A) = n$.

 Proof: $to - do$.

 Propesition:
 Let $A \in M_{p \times q} (\mathbb{F})$ and $B \in M_{q \times r} (\mathbb{F})$, then

 $\operatorname{vank} (A B) \leqslant \min \{ \operatorname{vank} (A), \operatorname{vank} (B) \}$

 Proof: Recall from midson exam that

 $\operatorname{van} (AB) \subseteq \operatorname{vois} (B)$
 $\Rightarrow \dim (\operatorname{vois} (AB)) \leqslant \dim (\operatorname{vois} (B))$

$$coi(AB) \subseteq col(A)$$

$$\Rightarrow dim(col(AB)) \leq dim(col(A))$$
So rank(AB) \leq min f rank(A), rank(B) f
$$\blacksquare$$
Full row rank: Let $A \in M_{mxn}(F)$, then A is said to be of full row rank if rank(A) = m.
Full column rank: Let $A \in M_{mxn}(F)$, then A is said to be of full column rank if rank(A) = m.

Proposition: Let A & Mmxn (IF), then the following are equivalent:

(a) A has full row rank.

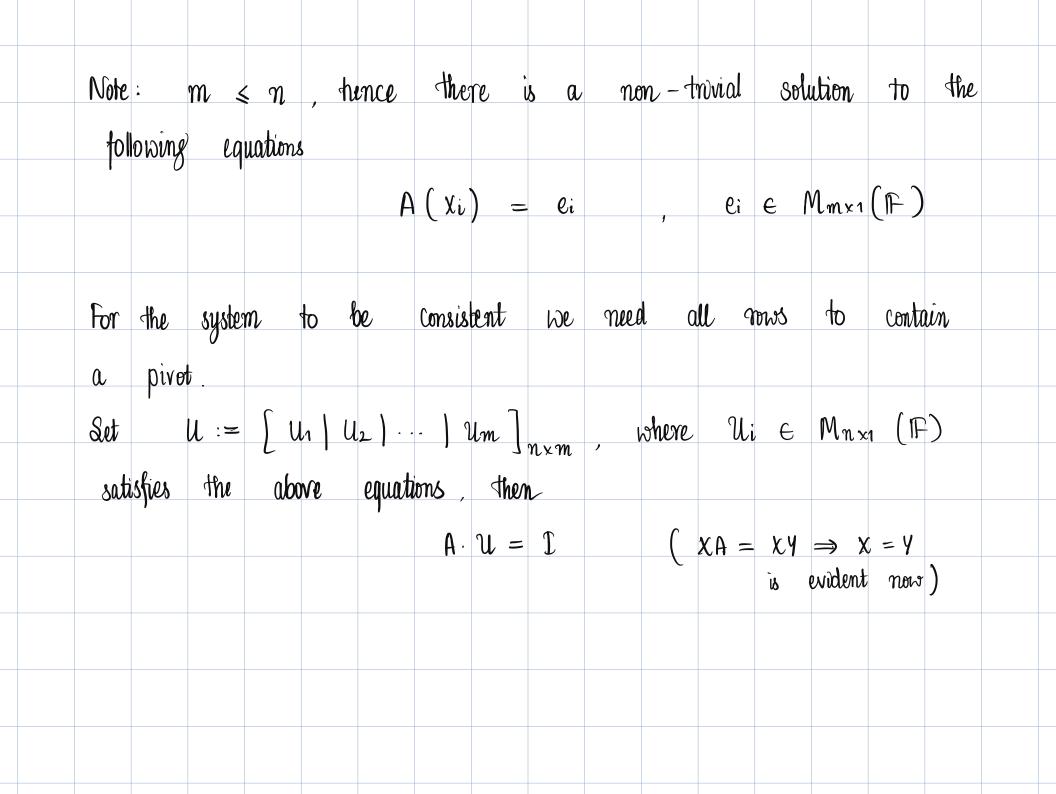
(b) A is night invertible, i.e., $XA = YA \implies X = Y$.

for appropriate matrices X and Y. (C) col(A) is $M_{m\times 1}$ (IF).

<u>Proof</u>: (a ⇒ b). Suppose A has full row rank, i.e., the row A_1 , ..., A_m of A are linearly independent in M_{1xn} (IF).

We claim there exists matrix $U \in M_{n \times m}$ (IF) such that

 $A_{mxn} U_{nxm} = I_{mxm} - (*)$



$$(b \Rightarrow c) \quad ket's \quad assume \quad that \quad \exists \ U \in M_{n\times m} \ (F) \quad st. \quad AU = U_{m\times m}$$
Then (c) is evident, as for the volumns U_i of U

$$We \quad have \qquad \sum_{j=1}^{n} U_{ji} \ (A_j) = e_i ; \quad vohere \quad e_i \in M_{m\times i} \ (F)$$
is the it vector of canonical ordered toasis
$$so \quad span \ (\ A_j \ i \ j = 1, ..., n \) = M_{m\times i} \ (F)$$

$$(c \Rightarrow a) \quad Suppose \quad span \ (\ A_j \ i \ j = 1, ..., n \) = M_{m\times i} \ (F)$$

$$since \quad column \ rank \quad is \quad same \quad as \quad rns \quad rank , the \ claimn \ holds$$

Note :	So	ution	to	the	r o	bove	syste	M	เช	not	uniqu)	In	particu	lar,	
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