2024/09/24 - Linear Algebra - Week 09

$W_1, W_2 \subseteq V$

$$\underbrace{W_1 + W_2}_{\equiv \{\text{span}} = \{ x + y \mid x \in W_1 , y \in W_2 \}$$

If for element $v \in W_1 + W_2$ $\exists$ unique $x \in W_1$ and $y \in W_2$ s.t. $v = x + y$, then $W_1 + W_2$ is said to be the direct sum of W_1 and W_2 and denoted by $\underbrace{W_1 \oplus W_2}_{\equiv \begin{cases} \text{span} + \\ \text{linear independence} \end{cases}}$.

⊛ $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Property: Let V be a vector space over $\mathbb{F}$ of finite dimension n, $n \geq 1$, and let $W_1, W_2 \subseteq V$ be subspaces of V

Then the following are equivalent:

(0) $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$

(1) $W_1 \cap W_2 = \{0\} = 0$

(2) $W_1 + W_2 = W_1 \oplus W_2$

(3) for any $x \in W_1 \setminus \{0\}$ and $y \in W_2 \{0\}$, x & y are linearly independent.

(4) $x \in W_1$, $y \in W_2$ then $x + y = 0 \Rightarrow x = 0$ and $y = 0$

Proof: $(0) \Longrightarrow (1)$

$\dim(W_1 \cap W_2) = 0$, so $W_1 \cap W_2 = \{0\}$

$(1) \Rightarrow (2)$

Assume: $x \in W_1$ $y \in W_2$

$x' \in W_1$ $y' \in W_2$

$$x + y = x' + y'$$

$$x - x' = y' - y \longrightarrow \circledast$$

LHS in $\circledast$ is in W_1

RHS in $\circledast$ is in W_2

$\because W_1 \cap W_2 = \{0\}$ $\therefore$ $x = x'$, $y = y'$

(2) $\Rightarrow$ (3)

every vector in $W_1 \oplus W_2$ has a unique representation

Let $ax + by = 0$, $a, b \in \mathbb{F}$

$\because$ $0 \in W_1 \oplus W_2$

0 has a unique representation

$0 = 0 \cdot x + 0 \cdot y$

~~So $a = 0$, $b = 0$ $\longrightarrow$ if not then 0 will have 2 representations~~

$\alpha_1 x + \alpha_2 y = 0$

0 has a unique

$0 = 0 + 0$

$ax \in W_1$

$by \in W_2$

$ax = 0$ and $by = 0$

$\Rightarrow$ $a = 0$ and $b = 0$

~~(3) $\Rightarrow$ (1)~~

~~Any $x \in W_1 \setminus \{0\}$, $y \in W_2 \setminus \{0\}$ is LI, then $W_1 \cap W_2 = \{0\}$~~

(3) $\Rightarrow$ (0)

Any $x \in W_1 \setminus \{0\}$, $y \in W_2 \setminus \{0\}$ is LI

then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$

If not, then $\dim(W_1 \cap W_2) \neq 0$

Let $x \in W_1 \cap W_2$

$x \in W_1 \setminus \{0\}$

$x \in W_2 \setminus \{0\}$

~~x is linearly dependent to x.~~

~~$x - x = 0$~~

$\{x, x\} \rightsquigarrow$ set unique objects

$\alpha x \in W_1 \cap W_2$ for any $\alpha \in \mathbb{F}$

$\{x, \alpha x\}$ $\alpha \in \mathbb{F} \setminus \{1\}$ is linearly dependent direct sum $\rightarrow$ was a binary operation

Define $W_1 \oplus W_2 \oplus W_3$, $W_1, W_2, W_3 \subseteq V$

$W_1 + W_2$

$(W_1 \oplus W_2) \oplus W_3$

<u>Guess</u>

$\dim(W_1 + W_2 + W_3)$

$= \dim(W_1) + \dim(W_2) + \dim(W_3)$

$- \dim(W_1 \cap W_2) - \dim(W_2 \cap W_3) - \dim(W_1 \cap W_3)$

$+ \dim(W_1 \cap W_2 \cap W_3)$.

subspaces $\equiv$ lattices

$\dim(W_1 + W_2) + \dim(W_3) - \dim((W_1 + W_2) \cap W_3)$

$W_1 \;, W_2 - (W_1 \cap W_2)$

problem set $\leftarrow$ intersection distributes over sum

$(W_1 + W_2) \cap W_3 \overset{?}{=} (W_1 \cap W_3) + (W_2 \cap W_3)$ $\boxed{\text{NO}}$

Rough

See problem set 10 & 11

x x

Direct sum ?

For any $x \in W_1 \setminus \{0\}$, $y \in W_2 \setminus \{0\}$, $z \in W_3 \setminus \{0\}$

then $\{x, y, z\}$ LI ? $W_1 + W_2 + W_3 = W_1 \oplus W_2 \oplus W_3$

$\boxed{\text{No}}$

See problem set 11

2×2 $\quad$ 2×1

$$A = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 5 \end{array}\right]_{3\times3}$$

1×2 $\quad$ 1×1

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]_{3\times3}$$

?

$$\left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]_{3\times3}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A_{21} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 5 \end{bmatrix}$$

Let $T \in L(V)$

Let $V = W_1 \oplus W_2 \oplus W_3$

$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$

$\mathcal{B}_1 = \{x_1, x_2, \dots, x_p\}$

$\mathcal{B}_2 = \{y_1, y_2, \dots, y_q\}$

$\mathcal{B}_3 = \{z_1, \cancel{z_2}, \dots, z_r\}$

$W_2 \times W_1$

$\mathcal{B}_1 \times \mathcal{B}_2$

$$
\begin{array}{c}
\begin{array}{ccccc} W_1 & \oplus & W_2 & \oplus & W_3 \end{array} \\
\begin{array}{ccc} T(x_1) \cdots T(x_p) & T(y_1) \cdots T(y_q) & T(z_1) \cdots T(z_r) \end{array}
\end{array}
$$

$$
[T]_{\mathcal{B} \times \mathcal{B}} = \left[\begin{array}{c|c|c}
[T_{11}] & [T_{12}]_{\mathcal{B}_1 \times \mathcal{B}_2} & [T_{13}] \\
\hline
[T_{21}] & [T_{22}] & [T_{31}] \\
\hline
[T_{31}] & [T_{32}] & [T_{33}]
\end{array}\right]
\begin{array}{c} W_1 \\ \oplus \\ W_2 \\ \oplus \\ W_3 \end{array}
$$

$[T_{11}]_{\mathcal{B}_1 \times \mathcal{B}_1}$ $\quad T$