2024/09/10 – Linear Algebra – Week 07

Recap:

Let V be a n- dimensional V.S and W be n - dimensional V.S $\Big\}$ over $\mathbb{F}$

$T \in L(V, W) \xrightarrow{\text{also}} \text{V.S}$

T can be represented by a $m \times n$ matrix with entries from $\mathbb{F}$

also V.S

linear map from $L(V, W)$ to $M_n(\mathbb{F})$

We know

Given any n-dimensional V.S V, $V \cong \mathbb{F}^n$

Example Coordinate space $\mathbb{R}^n$ over $\mathbb{R}$

Let $T \in L(\mathbb{R}^n)$

Basis of $\mathbb{R}^n$ — e_j $(0, 0, \ldots, \underset{j^{th}\text{ place}}{1}, 0, \ldots, 0)$

$$\mathcal{B} = \{\underbrace{e_j}_{\text{ordering } e_1, \ldots, e_n} : 1 \leq j \leq n\}$$

is the canonical ordered basis.

For any $v \in \mathbb{R}^n$

$$v = (v_1, v_2, \ldots, v_n) \qquad v_j \in \mathbb{R}$$

$$= \sum_{j=1}^{n} v_j e_j$$

$$T(v) = T\left(\sum_{j=1}^{n} v_j e_j\right)$$

$$T(v) = \sum_{j=1}^{n} v_j \cdot T(e_j)$$

[$\because$ T is a linear transformation it respects the linear structure]

changes $\longrightarrow$ changes

Let $f_j := T(e_j) \in \mathbb{R}^n$

knowing all f_j's is enough to know the transformation

transform any vector

$$T(v) = \sum_{j=1}^{n} v_j f_j$$

$$T(e_j) = \sum_{i=1}^{n} t_{ij} \, e_i \qquad 1 \leq j \leq n$$

$$T(e_1) = \sum_{i=1}^{n} t_{i1} \, e_1 \quad 1 \leq j \leq n$$

$$T(e_1) = (t_{11}, t_{21}, t_{31}, \dots, t_{n1})$$

$$T(e_2) = (t_{12}, t_{22}, \dots, t_{n2})$$

$$\vdots$$

$$T(e_j) = (t_{1j}, t_{2j}, \dots, t_{nj})$$

$$\boxed{T(e_j) = \sum_{i=1}^{n} t_{ij} \, e_i}$$

$$T(v) = \sum_{j=1}^{n} v_j \cdot T(e_j)$$

$$= \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{n} t_{ij} \, e_i \right)$$

$$= \sum_{j=1}^{n} \underbrace{\sum_{i=1}^{n} (\underset{\text{defines } T}{\underline{t_{ij}}} \cdot v_j)\, e_i}_{1 \text{ row of}}$$

To determine t_{ij}

$$\underbrace{\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & \vdots & \ddots & t_{2n} \\ \vdots & & & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}}_{\text{each column}} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \underbrace{\begin{bmatrix} \sum_{j=1}^{n} t_{1j} \cdot v_j \\ \sum_{j=1}^{n} t_{2j} \cdot v_j \\ \vdots \\ \sum_{j=1}^{n} t_{nj} \cdot v_j \end{bmatrix}}_{\text{dimension of } M_{n\times 1}(\mathbb{F}) = n} - (*)$$

is the coordinate of $T(e_j)$

$$\mathbb{R}^n \cong M_{n\times 1}(\mathbb{R})$$

$(*)$ $[T]\cdot[v] = [Tv]$

Matrix Representation : definition

Let V, W be two finite dimensional vector spaces over $\mathbb{F}$ with dimension $n, m \geqslant 1$ respectively.

Let $\mathcal{B}_1 = \{x_1, \ldots, x_n\}$

$\mathcal{B}_2 = \{y_1, \ldots, y_m\}$

be bases of V and W respectively.

Then the matrix representation of $T : V \to W$ is given by

$$[T]_{\mathcal{B}_2 \times \mathcal{B}_1} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix}$$

where $$T(x_j) = \sum_{i=1}^{m} t_{ij} y_i$$

2024/09/13

In spirit, all n-dimensional vector spaces are the same to us.

Theorem: Let V, W be vector spaces over $\mathbb{F}$ of dimension $n, m > 1$ respectively.
Then, the vector space $\mathcal{L}(V, W)$ is isomorphic to $M_{m \times n}(\mathbb{F})$

Proof:

$$\underset{n}{V} \xrightarrow{T} \underset{m}{W}$$

T corresponds to a $m \times n$ matrix

$$T \longleftrightarrow M_{m \times n}(\mathbb{F})$$

$$T \in \underset{\text{linear}}{L}\left(M_{n\times 1}(\mathbb{F}),\ M_{m\times 1}(\mathbb{F})\right)$$

$$T\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}$$

$$= T\left(\sum_{i=1}^{n}\alpha_i\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\right)$$

$$= \sum_{i=1}^{n}\alpha_i\, T\underbrace{\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}}_{e_j}$$

$$\left(T\underbrace{\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\end{bmatrix}}_{e_j} = \sum_{i=1}^{m} t_{ij}\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\right)$$

Particular

$$M_{n\times 1}(\mathbb{F}) \xrightarrow{T} M_{m\times 1}(\mathbb{F})$$

$$T\begin{bmatrix}|\end{bmatrix}_n \longrightarrow \begin{bmatrix}|\end{bmatrix}$$

$$T\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} t_{ij}\,\alpha_j\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\right)$$

$$\left[\begin{array}{c|c|c|c} \underbrace{\begin{matrix} t_{11}\\ t_{21}\\ \vdots\\ t_{m1}\end{matrix}}_{T\begin{bmatrix}1\\0\\0\\0\\ \vdots\end{bmatrix}} & \begin{matrix} t_{12}\\ t_{22}\\ \vdots\\ t_{m2}\end{matrix} & \cdots & \begin{matrix} t_{1n}\\ t_{2n}\\ \vdots\\ t_{mn}\end{matrix} \end{array}\right]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix} = T\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}$$

$V \;-\; \dim(V) = n$

over $\mathbb{F}$

$W \;-\; \dim(W) = m$

$$L(V, W) \cong M_{m \times n}(\mathbb{F})$$

<u>Proof</u>: Fix basis for V and W.

Let $\mathcal{B}_1$ be an ordered basis for V

$\mathcal{B}_2$ be an ordered basis for W

$\mathcal{B}_1 = \{x_1, \ldots, x_n\}$

$\mathcal{B}_2 = \{y_1, \ldots, y_m\}$

Let $T \in L(V, W)$

isomorphic to $M_{n \times m}(\mathbb{F})$ $\Longrightarrow$ <u>linear</u> bijection from $L(V, W)$ to $M_{m \times n}(\mathbb{F})$

$$T(x_j) = \sum_{i=1}^{m} t_{ij}\, y_i \qquad j = 1, \dots, n \longrightarrow (*)$$

Since $t_{ij} \in \mathbb{F}$ $\quad 1 \leq i \leq m, \quad 1 \leq j \leq n$

$$\Phi(T) = [T]_{\mathcal{B}_2 \times \mathcal{B}_1}$$

Claim: Φ is linear bijective map.

Let $T_1, T_2 \in L(V, W)$

① We need to prove:

$$[T_1 + \alpha T_2]_{\mathcal{B}_2 \times \mathcal{B}_1} = [T_1]_{\mathcal{B}_2 \times \mathcal{B}_1} + \alpha [T_2]_{\mathcal{B}_2 \times \mathcal{B}_1}$$

$$(T_1 + \alpha T_2)(x_j) = T_1(x_j) + \alpha \cdot T_2(x_j)$$

$$= \sum_{i=1}^{m} t_{ij}(x_j) + \alpha \sum_{i=1}^{m} t_{ij}^{(2)}(x_j)$$

$$= \sum_{i=1}^{m} (t_{ij} + \alpha \cdot t_{ij}^{(2)}) \cdot (x_j)$$

② Φ is one-one onto

<u>One one</u> $\Phi(T) = 0 \implies [T]_{\mathcal{B}_2 \times \mathcal{B}_1}$

From (*), $\Phi(T) = 0 \implies T(x_j) = 0 \quad \forall \quad j = 1 \ldots n$

Hence $T \equiv 0$

Onto Let $\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ t_{21} & & \vdots \\ \vdots & & \\ t_{m1} & & t_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{F})$

$$T(x_j) := \sum_{i=1}^{m} t_{ij}\, y_i \quad , \quad i = 1 \dots n$$

Extend T linearly

$$T\left(\sum_{j=1}^{n} \alpha_j x_j\right) := \sum_{j=1}^{m} \alpha_j\, T(x_j)$$

Verify T is linear

$\Phi : L(V, W) \longrightarrow M_{m\times n}(\mathbb{F})$ is an isomorphism

Example:

$I \in L(V)$ is the identity map on $\mathbb{R}^n$

$\mathcal{B}$ = basis of $\mathbb{R}^n$

Let $\mathcal{B}_1 = \{f_1, \dots, f_n\}$ and $\mathcal{B}_2 = \{g_1, \dots, g_n\}$ be two distinct bases of V

Then,
$$I(f_j) = f_j = \sum_{i=1}^{n} z_{ij}\, g_i$$

$$[I]_{\mathcal{B}_2 \times \mathcal{B}_1} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & & \vdots \\ z_{m1} & & \cdots & z_{mn} \end{bmatrix}$$

If $\mathcal{B}_1$ & $\mathcal{B}_2$ are not distinct, it can be easily seen that:

$$[I]_{\mathcal{B}_1 \times \mathcal{B}_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 1 & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example $\mathbb{R}^2$

$$I \in L(\mathbb{R}^2)$$

$$\mathcal{B}_1 = \{(1,0),\ (0,1)\}$$

$$\mathcal{B}_2 = \{(1,1),\ (1,-1)\}$$

$$[I]_{\mathcal{B}_1 \times \mathcal{B}_2} = ? \qquad [I]_{\mathcal{B}_2 \times \mathcal{B}_1} = ?$$

$\mathcal{P}_2(\mathbb{R})$

$$a_0 + a_1 x + a_2 x^2 \xrightarrow{D} a_1 + 2a_2 x$$

a_0, a_1, a_2

$\downarrow$

$a_1, 2a_2$

$$\mathcal{B} = \{1, x, x^2\}$$

$$[D]_{B \times B} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D(1) = 0 \cdot x^0 + 0 \cdot x + 0 \cdot x^2$$

$$D(x) = 1 \cdot x^0 + 0 \cdot x + 0 \cdot x^2$$

$$D(x^2) = 0 \cdot x^0 + 2 \cdot x + 0 \cdot x^2$$

$\downarrow$

$[D'']_{B \times B} =$

$\downarrow$

$[D''']_{B \times B} \rightsquigarrow$ nilpotent