2024/09/03 – Linear Algebra – Week 06

Linear transformation on vector spaces

→ map between two vector spaces over the same field.

→ preserves the linear structure.

0. Let $V_1 = \mathbb{R} = V_2$

$T(x) = 2x \quad , \quad x \in \mathbb{R}$

affine transformation

Let $a, b \in \mathbb{R} \quad T(x) = \boxed{\boxed{ax} + b} \quad , \quad x \in \mathbb{R}$

linear

For T to be linear, $b = 0$ → $\because T(0) = 0$

Examples:

1. Fix $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ then $T_\alpha : \mathbb{F}^n \to \mathbb{F}$ is defined by $T((v_1, \dots, v_n)) = \sum_{i=1}^{n} \alpha_i v_i$ is a linear transformation.

If $n = 1$, we get example 0.

Verification

$$T_\alpha((v_1, \dots, v_n) + \alpha (w_1, \dots, w_n)) \boxed{\overset{=}{\text{verify}}} T_\alpha(v_1, \dots, v_n) + \beta \cdot T_\alpha(w_1, \dots, w_n)$$

$\forall (v_1, \dots, v_n), (w_1, \dots, w_n) \in \mathbb{F}^n$

$\beta \in \mathbb{F}$

Let $T_\gamma : \mathbb{F}^n \longrightarrow \mathbb{F}$

$\gamma_1, \dots, \gamma_n \in \mathbb{F}$ fixed

$$T_\gamma((v_1, \dots, v_n)) = \sum_{i=1}^{n} \gamma_i v_i$$

$T : \mathbb{F}^n \longrightarrow \mathbb{F}^2$

$$T((v_1, \dots, v_n)) := \left(T_\alpha(v_1, \dots, v_n) , T_\gamma(v_1, \dots, v_n) \right)$$

linear $\equiv$ linear transformation

Is this T linear?

Suppose $n = 1$; α_1, γ_1

$T_\alpha : \mathbb{F} \to \mathbb{F}$ $\quad$ $T_\gamma : \mathbb{F} \to \mathbb{F}$

$T : \mathbb{F} \to \mathbb{F}^2$

$$T(x) = (\alpha_1 x, \gamma_1 x)$$

$$T(x + \beta w) = (\alpha_1 x + \alpha_1 \beta w , \gamma_1 x + \gamma_1 \beta w)$$

$T(x) + \beta T(w)$

$(\alpha_1 \cdot$

$=$

Today $\longrightarrow$ (Every finite dimensional V.S) $\longrightarrow$ is isomorphic to a coordinate space

$\longleftrightarrow$ bijection

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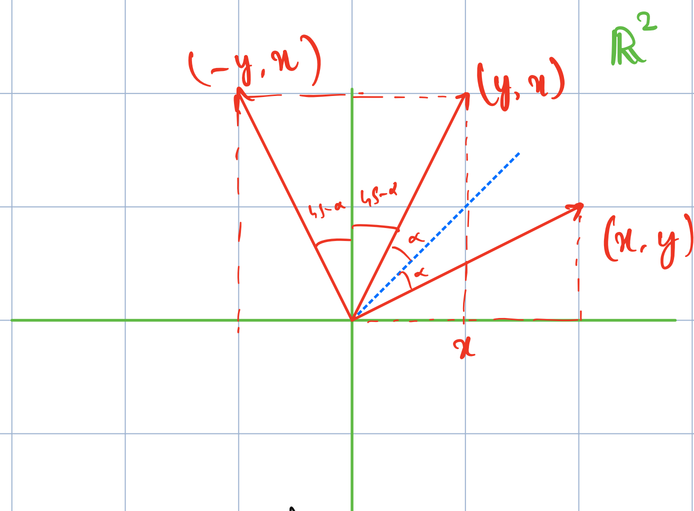
2. For $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T_1((x, y)) = (y, x)$$

$$T_2((x, y)) = (-y, x)$$



Zero map : $Z : V_1 \to V_2$

$$Z(v) = 0 \ , \quad \forall \ v \in V_1$$

Identity map: $I : V_1 \to V_2$

$$I(v) = v \quad \forall \ v \in V_1$$

~~coordinate~~ ~~code~~ for a graphical system

$\hookrightarrow$ linear transformation

stretching / squishing

shearing

all derivatives are linear transformation

5. $D \xrightarrow{} $ formal derivative

$$D : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$$

$$D(a_0 + a_1 x + a_1 x^2) = a_1 + 2a_2 x$$

6. $T : \mathcal{R}[0,1] \longrightarrow \mathbb{R}$

$$T(f) = \int_0^1 f(t)\,dt$$

$$\int (f+g) = \int f + \int g$$

$$\int c \cdot f = c \cdot \int f$$

7. $\mathrm{Re} : \mathbb{C} \longrightarrow \mathbb{R}$ $\quad \mathrm{Re}(x + iy) = x$

8. $\dfrac{d}{dt} : \mathcal{C}^1[0,1] \longrightarrow \mathcal{C}[0,1]$

$$f \longmapsto \frac{df}{dt}$$

9. $T : M_{2\times 1}(\mathbb{R}) \longrightarrow M_{3\times 1}(\mathbb{R})$

matrix multiplication distributes over the sum

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

need isomorphism for this

Every finite dimensional V.S can be represented as coordinate space

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Every LT can be represented as a Matrix

10. Let $A \in M_{m\times n}(\mathbb{F})$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Define $T : M_{n\times 1}(\mathbb{F}) \longrightarrow M_{m\times 1}(\mathbb{F})$

$$T(v) = A \cdot v \ , \quad v \in M_{n\times 1}(\mathbb{F})$$

Proposition

Let V, W be two vector spaces over $\mathbb{F}$, then the following statements hold

(1) $L(V, W)$ is a vector space over $\mathbb{F}$

w.r.t the following add^n on scalar multiplication

$$(T_1 + T_2)(v) := T_1(v) + T_2(v) \quad , \quad v \in V$$

$$(\alpha \cdot T_1) \cdot (v) := \alpha_1 \cdot T_1(v) \quad , \quad v \in V, \ \alpha \in \mathbb{F}$$

to be verified

(2) Composition of linear transformations

$\downarrow$ to be verified

$\downarrow$ product (is also a linear transformation)

$T_1 : V \to W$

$T_2 : W \to Z$

V, W, Z are all vector spaces over $\mathbb{F}$

$$\begin{tikzcd} V \arrow[r, "T_1"] \arrow[rr, bend right, "\underline{T_2 \circ T_1}"'] & W \arrow[r, "T_2"] & Z \end{tikzcd}$$

T_2 composed with T_1

(3) Let $T \in L(V)$ and T be an invertible function. Then T^{-1} is also a linear transformation

$\to$ T is one-one and onto

Claim

$$T^{-1}(v + \alpha w) = T^{-1}(v) + \alpha \cdot T^{-1}(w)$$

$\forall\ w, v \in V$ and $\alpha \in \mathbb{F}$

<u>Prove.</u>

$T(v) = y \qquad T(w) = x$

$T^{-1}(y) = v \qquad T^{-1}(x) = w$

$T(v + \alpha w) = y + \alpha \cdot x$

$T^{-1}(y + \alpha x) = v + \alpha w$

$T^{-1}(y) = v \qquad T^{-1}(x) = w$

Let $v, w \in V$ and $\alpha \in \mathbb{F}$

If $\left.\begin{array}{l} T^{-1}(v) = v_1 \\ \text{and } T^{-1}(w) = w_1 \end{array}\right\} \rightarrow T$ is onto

$$T\left(T^{-1}(v) + \alpha T^{-1}(w)\right) = v + \alpha w$$

Hence

$$T^{-1}(v + \alpha w) = T^{-1}(v) + \alpha T^{-1}(w) \qquad (\because T \text{ is one-one})$$

Every n-dimensional vector space $\xleftrightarrow[\text{one-one onto map}]{\text{isomorphic}}$ $\mathbb{F}^n$

Morph — structure

Morpheus — many faces

Matrix

Isomorphism: Let V_1 and V_2 be two finite-dimensional vector spaces over $\mathbb{F}$. A linear map $T \in L(V_1, V_2)$ is said to be an isomorphism if and only if T is a one-one onto map.

Moreover, if there exists an isomorphism between V_1 and V_2, then V_1 and V_2 are called isomorphic vector spaces.

Isomorphism Theorem

Let V be a n-dimensional vector space over $\mathbb{F}$.
Then V is isomorphic to the co-ordinate space $\mathbb{F}^n$.

Proof: Case-1: $n \geq 1$

Let V be an n-dimensional vector space over $\mathbb{F}$

Well Ordering Principle

Let $B = \{v_1, \ldots, v_n\}$ be an ordered basis

$B \to$ Linearly independent

spanning set

order

you can define T for every $v \in V$

Define a map:

$$T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = (\alpha_1, \ldots, \alpha_n)$$

$T \longrightarrow$ one - one & onto

Onto

Let $(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n$ then $\sum_{i=1}^{n} \alpha_i v_i \in V$

and by def^n $T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = (\alpha_1, \dots, \alpha_n)$

One - one: we need to verify if $T(v) = 0$ then $v = 0$

$$x_1 = x_2$$
$$\Updownarrow$$
$$f(x_1) = f(x_2)$$

$$x_1 - x_2 = 0$$
$$f(x_1) - f(x_2) = 0$$
$$\downarrow$$
$$LT \Rightarrow f(x_1 - x_2) = 0$$

Since B is basis, every $v \in V$ can be written as

$v = \sum_{i=1}^{n} \alpha_i v_i$; $T(v) = 0$ then $(\alpha_1, \dots, \alpha_n) = 0$

$\Rightarrow v = 0$

Definition of T depends on the basis and the ordering

- Canonical ordering

- Ask Problem set 2 (e) $\longrightarrow$ what are E & F ?

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Recap:

* Every n-dimensional vector space $n \geq 1$ over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

Proof:

V – n dimensional v.s over $\mathbb{F}$

Fix an ordered basis $B = \{v_1, \dots, v_n\}$

Any $x \in V$

$$x = \sum_{i=1}^{n} x_i v_i \qquad x_i \in \mathbb{F}$$

$$T(x) := (x_1, \dots, x_n)$$

T is one-one, T is onto $\longrightarrow$ $\forall\ (v_1, \ldots, v_n) \in T$ exist $v \in V$

and T is linear

$V \cong F$

Corollary:

Any two finite dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Proof:

$$V \overset{T_1}{\longleftrightarrow} \mathbb{F}^n \qquad \Longrightarrow \qquad \underbrace{V \longleftrightarrow W}_{\text{composition}} \quad T_1(T_2^{-1})$$
$$W \overset{T_2}{\longleftrightarrow} \mathbb{F}^n$$

Suppose

V is isomorphic to $\mathbb{F}^n$ via T_1

i.e., $T_1 : V \to \mathbb{F}^n$, T_1 is linear bijection

W is isomorphic to $\mathbb{F}^n$ via T_2

$$V \xrightarrow{T_1} \mathbb{F}^n \underset{T_2}{\overset{T_2^{-1}}{\rightleftarrows}} W$$

$$T_2^{-1} \cdot T_1 : V \to W$$

and $T_2^{-1} \cdot T_1$ is a linear bijection

basis change
$\Downarrow$
address of vector changes (vector itself is the same)

Example:

$\mathcal{P}_2(\mathbb{R})$, $\dim(\mathcal{P}_2(\mathbb{R})) = 3$

isomorphic to $\mathbb{R}^3$

$\mathcal{P}_2(\mathbb{R}) \rightsquigarrow$ canonical basis (canonically ordered)

$= \{1, x, x^2\}$

$\mathbb{R}^3 \longrightarrow$ canonical basis

$$= \{ (0,0,1), (0,1,0), (1,0,0) \}$$

$$a_0 + a_1 x + a_2 x^2 \longleftrightarrow (a_0, a_1, a_2)$$

$$1 + x \longleftrightarrow (1, 1, 0)$$

$$1 - x^2 \longleftrightarrow (1, 0, -1)$$

$$1 + x + x^2 \longleftrightarrow (1, 1, 1)$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

col(A) over $\mathbb{R}$

span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $M_{3\times1}(\mathbb{R})$

$\rightarrow \quad M_{3\times 1}(\mathbb{R}) \cong \mathbb{R}^3$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \longleftrightarrow (1, 1, 0) \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longleftrightarrow (1, 1, 1)$$

$\approx 1+x \qquad \approx 1 + x + x^2$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \longleftrightarrow (1, 0, -1)$$

$\approx 1 - x^2$

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

(only)
check if A is invertible or not

find determinant

$\det(A) = -1$

You don't need to solve for the entire system

$\longrightarrow$ use the corollary to form A

find $\det(A)$.

Exercise.

1. Consider the vector space $\mathcal{P}_2(\mathbb{R})$ with a basis given by

$\mathcal{B} = \{1 + x,\ 1 - x^2,\ 1 + x + x^2\}$

ordered basis

order $\longrightarrow$ $v_1 = 1+x$ $\quad$ $v_3 = 1 + x + x^2$

$v_2 = 1 - x^2$

What is the coordinate representation of $\mathcal{P}_2(\mathbb{R})$ with respect to the above ordered basis?

$$p = a_0 + a_1 x + a_2 x^2 = \alpha_1(1+x) + \alpha_2(1-x^2)$$
$$+ \alpha_3(1+x+x^2)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 1 & 0 & 1 & a_2 \\ 0 & -1 & 1 & a_3 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & a_1 \\ 0 & -1 & 0 & a_2 - a_1 \\ 0 & -1 & 1 & a_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & a_2 \\ 0 & -1 & 0 & a_2 - a_1 \\ 0 & -1 & 1 & a_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & a_2 \\ 0 & 1 & 0 & a_1 - a_2 \\ 0 & 0 & 1 & a_1 - a_2 + a_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2a_2 - a_1 - a_3 \\ 0 & 1 & 0 & a_1 - a_2 \\ 0 & 0 & 1 & a_1 - a_2 + a_3 \end{array}\right]$$

$2a_1 - a_0 - a_2$

$a_0 - a_1$

$a_0 - a_1 + a_2$

$$\cancel{2}a_2 - \cancel{a_1} - \cancel{a_3} + \cancel{a_1} - \cancel{a_2} + a_1 - \cancel{a_2} + \cancel{a_3}$$

$$\cancel{2}a_2 - \cancel{a_1} - \cancel{a_3} + \cancel{a_1} - \cancel{a_2} + \cancel{a_3}$$

$$\cancel{a_1} - \cancel{a_2} + a_3 + \cancel{a_2} - \cancel{a_1}$$

2. Find the coordinate representation of $\mathbb{R}^3$ with respect to the ordered bases: (i) $\mathcal{B}_1 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$

(ii) $\mathcal{B}_2 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$

$$\mathcal{B}_2 = \{(0,0,1),\ (0,1,0),\ (1,0,0)\}$$

$$(\alpha_0, \alpha_1, \alpha_2) \xleftrightarrow{\mathcal{B}_2} (\alpha_2, \alpha_1, \alpha_0)$$

Q: How do we relate two distinct co-oridinate representation of a finite dimensional v.s?

Matrix Representation of a linear transformation

Let V, W be vector spaces over $\mathbb{F}$. Then $L(V, W)$ is also a vector space over $\mathbb{F}$.

What is the dimension of $L(V, W)$ given V, W are finite dimensional?

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be ordered basis of V and W resp.

$\dim(V) = n$

$\dim(W) = m$

$\mathcal{B}_1 = \{x_1, x_2, \cdots, x_n\}$

$\mathcal{B}_2 = \{y_1, \cdots, y_m\}$

$T(v) \longrightarrow$ can get from $T(x_i)$ $\swarrow$ linear comb

Let $T \in L(V, W)$

$T(x_j) \in W$ for each $j = 1, \ldots, n$

each

$$\boxed{T(x_j) = \sum_{i=1}^{m} \alpha_{ij} \, y_i} \text{ --- } (*)$$

Claim : $T \underset{\text{isomorphic}}{\longleftrightarrow} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{12} & \cdots & \alpha_{2n} \\ \vdots & & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}$

$$\Phi(T) = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{m \times n} \end{bmatrix}, \quad T \in L(V, W)$$

where α_{ij} satisfy $*$

$$\Phi : L(V, W) \longrightarrow M_{m \times n}(\mathbb{F})$$

<u>Claim</u>: Φ is an isomorphism.

$$
\begin{tikzcd}
V \arrow[r, leftrightarrow] \arrow[d] & \mathbb{R}^n \arrow[d, "{[\alpha_{ij}]}"] \\
W \arrow[r, leftrightarrow] & \mathbb{R}^m
\end{tikzcd}
$$

$$v \longleftrightarrow (v_1, \ldots, v_n)$$

then $$T(v) \longleftrightarrow \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$