2024/08/20 - Linear Algebra - Week 04

Null space

Let $A \in M_{m\times n}(\mathbb{F})$ be any matrix, then the null space of A is defined by

$$\mathrm{Null}(A) = \{ X \in M_{n\times 1}(\mathbb{F}) \mid A(X) = 0 \}$$

In other words, null space is the solution space of a system of homogeneous linear equations. [Pg 36, HK]

$\mathrm{Null}(A)$ is a subspace, $\mathrm{Null}(A) \subseteq M_{n\times 1}(\mathbb{F})$

Verify: (i) $0 \in \mathrm{Null}(A)$ $\because$ $A\begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} = 0$

(ii) Let $f, g \in \mathrm{Null}(A)$ and $\alpha \in \mathbb{F}$

then $f + \alpha\cdot g \in \mathrm{Null}(A)$?

$$\begin{aligned} A\cdot(f) &= 0 \\ A\cdot(g) &= 0 \end{aligned} \quad \Rightarrow \quad A(\alpha\cdot f + g) = 0$$

* $\{[x_1, x_2, x_3]^T \mid x_1 + x_2 - x_3 = 0\} = \text{Null}(A)$

for some A.

Find A.

$A(x) = 0$

$\underset{\substack{\downarrow \\ 1 \text{ eq}^n}}{m \times n} \quad \underset{\substack{\downarrow \\ 3}}{n \times 1}$

$$[1, \ 1, \ -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

new subspaces from older ones $\longrightarrow$ Subspace calculus

Proposition

Let V be any vector space over $\mathbb{F}$, and let W_1, W_2 be two subspaces of V. Then $W_1 \cap W_2$ is also a subspace of V.

Proof:

We need to verify: (i) $0 \in W_1 \cap W_2$ $\quad$ ($\because$ $0 \in W_1$ and $0 \in W_2$)

(ii) for any $v, w \in W_1 \cap W_2$ and

$\alpha \in \mathbb{F}$, $\quad v + \alpha w \in W_1 \cap W_2$

$i = 1, 2$ if $v, w \in W_i$, $\alpha \in \mathbb{F}$

$$\Rightarrow \quad v + \alpha w \in W_i$$

Hence, (ii) holds.

More generally: Let V be a vector space over field F. The intersection of any collection of subspaces of V is a subspace of V. [Pg 36, HK]

$$\boxed{f + \alpha g \in \text{subspace}} + \boxed{\text{Non-empty}}$$

$$\equiv$$

$$\boxed{f + \alpha g \in \text{subspace}} \quad \boxed{0 \in \text{subspace}}$$

$A \in M_n(\mathbb{F})$ is an upper triangular matrix if the $(i, j)^{th}$ entry of A satisfies the following:

$$A_{ij} = 0 \quad \text{if} \quad i > j$$

* $L_n(\mathbb{F})$ and $U_n(\mathbb{F})$ are subspaces of $M_n(\mathbb{F})$

$\Rightarrow$ $L_n \cap U_n \equiv$ diagonal matrix is a subspace

Spanning Set

Let V be a vector space over $\mathbb{F}$. Let $S \subseteq V$ ($\neq \phi$) be any subset of V. We say that S is a spanning set for V if for any given $v \in V$, $\exists$ finitely many elements $s_1, \dots, s_n \in S$ such that

$$v = \sum_{i=1}^{n} \alpha_i s_i \quad \text{for some } \alpha_1, \dots, \alpha_n \in \mathbb{F}$$

Example

Consider the vector space $\mathbb{R}$ over $\mathbb{R}$.

$S = \{1\}$ is a spanning set for $\mathbb{R}$

$S = \{\alpha\}$, $\alpha \neq 0$ is a spanning set for $\mathbb{R}$

Note: Spanning sets may or may not be finite.

example: polynomial space

Example: For $\mathbb{R}^2$, $\{(1, 1), (1, -1), (0, 1)\}$ is a spanning set.

* Take $(v_1, v_2) \in \mathbb{R}^2$

Show that $\exists\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

such that

$$(v_1, v_2) = \alpha_1(1,1) + \alpha_2(1,-1) + \alpha_3(0,1)$$

$$v_1 = \alpha_1 + \alpha_2$$

$$v_2 = \alpha_1 - \alpha_2 + \alpha_3$$

Finding $\{\alpha_1, \alpha_2, \alpha_3\} \equiv$ finding solution for the above system.

<u>Alternatively</u>

$\alpha_1 = 0$

$\alpha_2 = v_1$

$\alpha_3 = v_2 + v_1$

* Using RREF, general solution:

$$\alpha_1 \in \mathbb{R}, \quad \alpha_2 = v_1 - \alpha_1$$

$$\alpha_3 = v_2 + v_1 - 2\alpha_1$$

<u>Observation</u>

* For a spanning set, the linear combination to form a vector need not be unique.

* The α_i's in the above definition is not unique. That is, it is possible for $v \in V$,

$\exists s_1, s_2, \dots, s_n \in S$ and $s_1', s_2', \dots, s_m' \in S$ such that

$$v = \sum_{i=1}^{n} \alpha_i s_i = \sum_{j=1}^{m} \alpha_i' s_j'$$

$\alpha_i \in \mathbb{R}$, $\alpha_i' \in \mathbb{R}$

<u>Example</u>: * The set $S = \{(1,1), (-1,1)\}$ spans over $\mathbb{R}^2$.

* $\mathbb{C}$ is a vector space over $\mathbb{R}$. Then $\{1, i\} \subseteq \mathbb{C}$ is a spanning set for $\mathbb{C}$ over $\mathbb{R}$.

* But $\{1\}$ or $\{i\}$ is not a spanning set for $\mathbb{C}$ over $\mathbb{R}$. It is a spanning set for $\mathbb{C}$ over $\mathbb{C}$

$i \cdot i = -1$

* For $A \in M_{n \times m}(\mathbb{F})$, a spanning set for the space $\mathrm{col}(A)$ is the set of all columns of A.

<u>Span of a set</u>

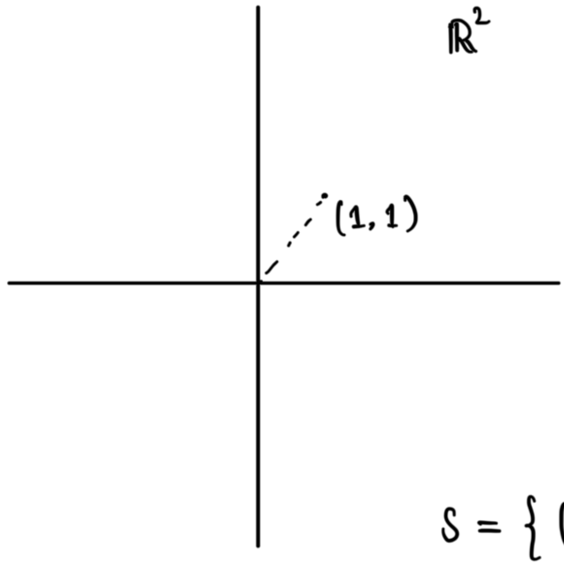
Let V be a vector space over $\mathbb{F}$. Let $S\,(\neq \phi) \subseteq V$

Then, the span of S is

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{n} \alpha_i s_i \;\middle|\; s_i \in S,\ \alpha \in \mathbb{F},\ n \in \mathbb{N} \right\}$$

* For any $S\,(\neq \phi) \subseteq V$, $\mathrm{span}(S)$ is subspace of V

* span (S) is also called the subspace generated by S.



$$S = \{(1,1)\}$$

$$\text{span}(S) = \{(x, y) \mid x = y\}$$

$$= \{\alpha (1,1) \mid \alpha \in \mathbb{R}\}$$

$$S = \{(0,1), (1,0)\}$$

$$\text{span}(S) = \mathbb{R}^2$$

Next class : Dimension and basis of a vector space.