2024/08/13 – Linear Algebra – Week 03

Vector Spaces

Subspaces

Let V be a vector space over $\mathbb{F}$ with addition '$+$' and scalar multiplication '$\cdot$'. Then a subset $W (\neq \phi) \subseteq V$ is said to be a subspace of V, if W is a vector space over $\mathbb{F}$ with the same addition '$+$' and scalar multiplication '$\cdot$'.

check closeness for W

Remark: Let V be a vector space over $\mathbb{F}$ and $W \subseteq V$ be non-empty. W is a subspace of V if and only if

(a) $0 \in W$

(b) $f, g \in W \Rightarrow f + \alpha \cdot g \in W \quad \forall \alpha \in \mathbb{F}$

Examples:

1. Co-ordinate space $\mathbb{R}^3$

Following are some natural subspaces of $\mathbb{R}^3$

(i) $P_{xy} = \{ (x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R} \}$

|||ly P_{yz}, P_{zx} (YZ plane, XZ planes) are subspaces

(a) $0 \in P_{xy}$

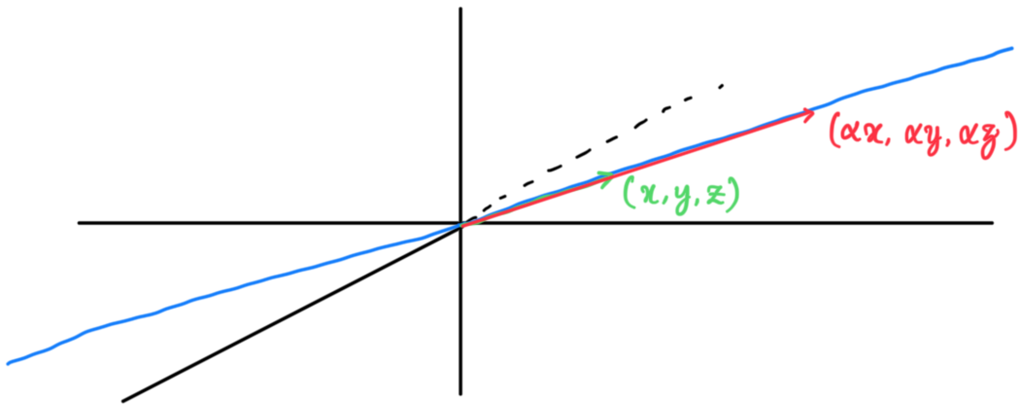
(b) $(x_1, x_2, 0) + \alpha (x_1, x_2, 0)$

(ii) x - axis :- $L_{(1,0,0)} = \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\} = (x_1 + \alpha x_1,$

$x_2 + \alpha x_2, 0)$

Trivial spaces $\left\{ \right.$ (iii) Zero space ~~Null space~~ (iv) $\mathbb{R}^3$

(v) any line or plane passing through the origin



$(a, b, c) \in \mathbb{R}$

$$P_{(a,b,c)} = \{(x_1, x_2, x_3) \mid ax_1 + bx_2 + cx_3 = 0\}$$

plane passing through origin

$(a^2 + b^2 + c^2 \neq 0)$

space of Riemann integrable f^ns is bigger than $\mathcal{C}[0,1]$

2. The space of all continuous functions $\mathcal{C}[0,1]$ with respect to usual addition of functions and multiplication by scalars is a subspace of $R[0,1]$, the space of all

Riemann integrable functions over $\mathbb{R}$

3. Both $R[0,1]$ and $\mathcal{C}[0,1]$ are subspaces of $\mathbb{R}^{[0,1]}$, the space of all real valued functions over $\mathbb{R}$.

4. For every vector space V over $\mathbb{F}$, the space V and $\{0\}$ are always subspaces. These are called trivial subspaces.

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5. Row space and column space

$M_{m\times 1}(\mathbb{F})$

$M_{1\times n}(\mathbb{F})$

Let $A \in M_{m\times n}(\mathbb{F})$. Consider the following set:

$$\mathrm{col}(A) = \{ A\cdot X \mid X \in M_{n\times 1}(\mathbb{F})\}$$

Clearly, $\mathrm{col}(A) \subseteq M_{m\times 1}(\mathbb{F})$

<u>Note</u>: if $f \in \mathrm{col}(A)$ and $g \in \mathrm{col}(A)$, then

$\exists$ $x \in M_{n\times 1}(\mathbb{F})$ and $y \in M_{n\times 1}(\mathbb{F})$ such that

$$A(x) = f \quad \text{and} \quad A(y) = g$$

And for any $\alpha \in \mathbb{F}$,

$$\begin{aligned} f + \alpha\cdot g &= A(x) + \alpha\cdot A(y) \\ &= A(x + \alpha y) \end{aligned}$$

So, $(f + \alpha \cdot g) \in col(A)$.

Also, $0 \in col(A)$ $\quad [\because A(0) = 0]$

Hence, $col(A) \subseteq M_{n \times 1}(\mathbb{F})$ is a subspace

$$A \cdot X = [A_1 | A_2 | \dots | A_n] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \underbrace{x_1 \cdot A_1 + x_2 \cdot A_2 + \dots + x_n A_n}_{\text{linear combination of columns}}$$

Multiplying A with a column matrix $\equiv$ getting a linear combination of its columns

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\operatorname{col}(A) = \cancel{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}}$$

Bro, it's called column matrix for a reason 😔

$$\operatorname{col}(A) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot X \;\middle|\; X \in M_{2\times 1}(\mathbb{F}) \right\}$$

$$= x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\operatorname{col}\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right) = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \cdot X \;\middle|\; X \in M_{2\times 1}(\mathbb{F}) \right\}$$

$$= \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 \\ x_1 + 2x_2 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 1 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

Row space

Let $A \in M_{m \times n}(\mathbb{F})$. Then the row space of A is defined to be

$$\text{row}(A) = \{ x \cdot A \mid x \in M_{1 \times m}(\mathbb{F}) \}$$

Clearly, $\text{row}(A) \subseteq M_{1 \times m}(\mathbb{F})$

Note: if $f \in \text{row}(A)$ and $g \in \text{row}(A)$, then

$\exists x, y \in M_{1 \times m}(\mathbb{F})$ such that $x(A) = f$ and $y(A) = g$.

$$f + \alpha \cdot g = x(A) + \alpha y(A)$$
$$= (x + \alpha y) A$$

So, $(f + \alpha \cdot g) \in \text{row}(A)$

Also, $0 \in \text{row}(A)$ $\quad [\because A(0) = 0]$

Hence, $\text{row}(A) \subseteq M_{1 \times m}(\mathbb{F})$ is a subspace.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{1 \times m} \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{m \times n}$$

$$[a + d + g \quad b + e + h$$

$$x \cdot A = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$$

$$= x_1 \cdot$$

Remarks:

1. $f \in col(A) \iff f^T \in row(A^T)$

$$f = A \cdot X$$
$$f^T = (AX)^T = X^T A^T$$

Trivial

2. Let $A \in M_{m \times n}(\mathbb{F})$, then the system of linear equations

$$A(x) = b$$

has a solution if and only if $b \in col(A)$

Let A be a matrix in RREF and

$A_{P_1}, A_{P_2}, \ldots, A_{P_r}$ be the basic columns

of A, then

$$A_q \in col(A) \quad , \quad q \in \{1, \ldots, n\} \setminus \{P_1, \ldots, P_r\}$$

Proof: Without loss of generality,

assume $r < n$.

Let A_q be any free column of A.

Suppose $A_q = \begin{bmatrix} b_{1q} \\ b_{2q} \\ \vdots \\ b_{mq} \end{bmatrix}$

$b_{iq} = 0$ whenever $i > r$

Hence, $A_q = \begin{bmatrix} b_{1q} \\ \vdots \\ b_{rq} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Clearly,

$$A_q = \sum_{i=1}^{r} b_{iq} A_{p_i}$$

Let $X \in M_{n \times 1}(\mathbb{F})$ such that

$$X_{p_i} = b_{iq} \qquad \text{for } i = 1, \ldots, r$$

$$X_j = 0 \qquad \text{for } j \neq p_i$$

then $A \cdot X = A_q$

$\square$