2024/08/06

Linear Algebra – Week 2

System of equations $\longrightarrow$ nice form $\longrightarrow$ Gaussian elimination

operations on augmented matrix $\}\mapsto$ elementary row (+ column) operations

Elementary Matrices

A matrix is said to be row-elementary if it is formed by performing row-operations on the identity matrix.

Consider the identity matrix of size 3×3

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftarrow \alpha \cdot R_1]{② \ \alpha \neq 0} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I \xrightarrow{① \ R_1 \longleftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$I \xrightarrow{R_1 \leftarrow R_1 + \alpha R_3} \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Row)

Echelone form :

$A \in M_{m\times n}(\mathbb{F})$ is said to be in row-echelon form if the following are satisfied:

(1) The r non-zero rows are above the $(m-r)$ zero rows

(2) If the first non-zero entry of the i^{th} row is at the p_i^{th} place, then

$$p_1 < p_2 < \cdots < p_r$$

(3) $a_{ip_i} = 1$ for $i = 1, \ldots, r$

A typical matrix in row-echelon form looks like

$$\begin{array}{cc} r \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \\ (m-r)\ \{ & \end{array} \begin{bmatrix} 0 & 1 & * & * & \cdots & * \\ 0 & 0 & 0 & 1 & \cdots & * \\ 0 & 0 & 0 & 0 & 1 \ \cdots & * \\ 0 & 0 & 0 & 0 & 0 \ \cdots & 0 \end{bmatrix}$$

$m-r$ can be zero

<u>Example</u>

1

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 3 & 0 & 2 & -1 \\ 1 & 5 & 0 & 8 \end{bmatrix}$$

$\downarrow$ $R_1 \to \frac{1}{2} R_1$ $\quad$ $\frac{1}{2} R_1 \xrightarrow[\text{replaces}]{} R_1$

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 3 & 0 & 2 & -1 \\ 1 & 5 & 0 & 8 \end{bmatrix}$$

$$\Big\downarrow \; R_2 - 3R_1 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 0 & -9/2 & \left(3/2+2\right) & -1 \\ 1 & 5 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 0 & -9/2 & 7/2 & -1 \\ 1 & 5 & 0 & 8 \end{bmatrix}$$

$$\Big\downarrow \; -R_1 + R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 0 & -9/2 & 7/2 & -1 \\ 0 & 7/2 & 1/2 & 8 \end{bmatrix}$$

$$\Big\downarrow \; R_2\left(\frac{-2}{9}\right) \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 0 & 1 & -7/9 & 2/9 \\ 0 & 7/2 & 1/2 & 8 \end{bmatrix}$$

$$\Big\downarrow \; \frac{-7}{2} R_2 + R_3 \longrightarrow R_3$$

$$\vdots$$

Echelone forms are not unique

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 0 \\ 0 & 1 & -7/9 & 2/9 \\ 0 & 0 & 1 & 65/29 \end{bmatrix}$$

- [x] Algorithm – Given a matrix find an echelon form (computational linear algebra)

  Linear Algebra – Week...duced Row Echelon Form

* Row-echelon forms are not unique, but the number of pivot elements are unique

Using echelon forms to solve a system of linear equations

$$\begin{aligned} x + y + z &= 6 \\ 2x + 2y + 3z &= 14 \\ 3x + 3y + 4z &= 20 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 2 & 3 & 14 \\ 3 & 3 & 4 & 20 \end{array}\right] \xrightarrow[-R_2 + R_3 \to R_3]{\substack{-2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x + y + z &= 6 \\ z &= 2 \end{aligned}\right\} \to \text{(System I)}$$

$$\left.\begin{aligned} x + y &= 4 \\ z &= 2 \end{aligned}\right\} \to \text{(System II)}$$

General solution

Solution set:

$$\boxed{x = 4 - y \qquad (x \leftrightarrow \text{pivot})}$$

$S := \{[4-y,\ y,\ 2]^T : y \in \mathbb{F}\}$ $\boxed{z = 2 \qquad (z \leftrightarrow \text{pivot})}$

Alternatively, $S := \{ [4,\ 0,\ 2]^T + \alpha[-1,\ 1,\ 0]^T,\ \alpha \in \mathbb{R}\}$

Putting $\alpha = \text{some value}$ gives you a particular solution

$\alpha = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 20 \end{bmatrix}$$

<u>Notice</u>

(System I) (System II)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_2 + R_1 \to R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced-row echelone form

<u>Reduced Row-Echelon form</u>

A matrix $A \in M_{m \times n}(\mathbb{F})$ is said to be in a reduced row-echelon form if the corresponding column to every pivot element a_{ip_i} of a non-zero row R_i is

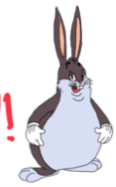
$$e_{p_i} := [\,0,\ 0,\ \dots,\ \underset{i^{th}}{1},\ 0,\ \dots\ 0\,]^T$$

For $A \in M_{m \times n}(\mathbb{F})$, its reduced row echelon form is denoted by E_A

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 4 & 6 & 2 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix}$$

Always use the pivot element !!!

f** ked



$$\Big\downarrow\ R_1 - R_4 \to R_1$$

$$\begin{bmatrix} 0 & 0 & -2 & -2 & -2 \\ 2 & 4 & 4 & 6 & 2 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix}$$

$$\Big\downarrow\ R_2 - 2R_4 \to R_2$$

$$\begin{bmatrix} 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & -4 & -4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix}$$

$$\Big\downarrow \; R_1/_{-2} \to R_1 \;\oplus\; R_2/_{-4} \to R_2$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix}$$

✓ add clown emoji here. 🤡

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 1 & 4 \end{array}\right]$$

$$\Big\downarrow \; -2R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & -2 \\ 3 & 6 & 1 & 4 \end{bmatrix}$$

$$\Big\downarrow \; -R_2/2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 1 & 4 \end{bmatrix} \xrightarrow[\rightarrow R_3]{-3R_1 + R_3} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\Big\downarrow$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\boxed{\text{Echelon}}$

$$\Big\downarrow$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced echelon

$$\begin{cases} x + 2y = 1 \\ z = 1 \end{cases}$$

$x = 1 - 2y$ ($x \Leftrightarrow$ pivot)

$z = 1$

$$S := \{ [1 - 2y, \; y, \; 1]^T \;, \; y \in \mathbb{F} \}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 2 & 0 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{matrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 2 & -2 & 2 \end{matrix}$$

no solution ← ~~(echelon form cannot be constructed)~~

inconsistent system

lazy ☹

Echelon form

$$E_4 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \} \; 0x + 0y + 0z = 1$$

swapping

$R_i \leftrightarrow R_j \quad : \quad E_{ij}$

$\alpha \cdot R_i \longrightarrow R_i \quad : \quad E_i(\alpha)$

$$\alpha \cdot R_j + R_i \to R_i \quad : \quad E_{ij}(\alpha)$$

① Elementary row operations $\equiv$ pre-multiplying elementary matrices

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\|$ verify

$$\alpha R_3 + R_2 \longrightarrow R_2$$

$$A_\alpha := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \alpha a_{31} & a_{22} + \alpha a_{32} & a_{23} + \alpha a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

<u>2024/08/09</u>

② Each elementary matrix is invertible.

Elementary row operations are pre-multiplications with invertible matrices

$$E_{ij} \cdot E_{ij} = I$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E_{13}} \cdot \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E_{13}} = I$$

$$\alpha R_3 + R_2 \longrightarrow R_2$$

$$? \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Observation

Let A be an $m \times n$ matrix then A can be written in the RREF by pre-multiplying a finite sequence of elementary matrices.

Let A be invertible.

$$A \cdot A^{-1} = I$$

$$A \cdot \left[A_1^{-1} \mid A_2^{-1} \mid A_3^{-1} \mid \cdots \mid A_n^{-1} \right] = I$$

$$\left[A(A_1^{-1}) \mid A(A_2^{-1}) \mid \cdots \mid A(A_n^{-1}) \right] = \left[e_1 \mid e_2 \mid \cdots \mid e_n \right]$$

$$A(A_i^{-1}) = e_i$$

$$A \cdot X_i = e_i \qquad \text{for} \quad i = 1, \ldots, n$$

$[A \mid e_i] \longrightarrow$ augmented matrix for i^{th} equation

n eq^n in 1 augmentation

$[A \mid I]$

$[A \mid (e_1 \mid e_2 \mid \cdots \mid e_n)]$

Claim: unique RREF of an invertible matrix $= \underline{I}$

$\Rightarrow$ maximum pivots in RREF

$$\left[\, A \mid (e_1 \mid e_2 \mid \cdots \mid e_n) \,\right]$$
$$\Big\downarrow \text{ elementary operations}$$
$$\left[\, I \mid B_1 \mid B_2 \mid \cdots \mid B_n \,\right]$$

$$A^{-1} = \left[\, B_1 \mid B_2 \mid \cdots \mid B_n \,\right]$$

Theorem:

Let $A \in M_{n \times m}(\mathbb{F})$ be an invertible matrix. Let its RREF be E_A

Then, $E_A = I$

Proof: Let $E_1, E_2, \ldots, E_k$ be the elementary matrices that correspond to the elementary operations $e_1, e_2, \ldots, e_k$ on A which yield E_A (RREF of A).

$$E_1 \cdot E_2 \cdot \cdots \cdot E_k \cdot A = E_A$$

Now, since each elementary matrix is invertible and A is given to be invertible, hence matrix on the LHS is invertible

i.e., E_A is invertible.

Since E_A is invertible, it cannot have any zero rows.

Hence, by the properties of RREF, E_A has to be I.

Proof

Introduction to Vector Spaces and Linear Transformations

Consider the following RREF

$$E_A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let C_1, C_2, C_3, C_4 be the columns of E_A

$$2 \cdot C_1 = C_2 \qquad 3C_1 + 4C_3 = C_4$$

So, column 2 and column 4 can be written by combining column 1 and column 3 with some scalar factor

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Definition

$\underset{\sim}{\text{binary}}$ $\mathbb{Z}_2, \mathbb{R}, \mathbb{C}$

Vector → magnitude & direction
→ abstract (here)

Let $\mathbb{F}$ be a field.

$$(V, \mathbb{F}, +, \cdot)$$

is said to be a vector space over $\mathbb{F}$ if the following hold:

① V has vector addition

(i) $x + y = y + x$

(ii) $(x + y) + z = x + (y + z)$

(iii) $\exists\, 0 \in V \;(x + 0 = x \quad \forall x \in V)$

(iv) $\forall x \in V \left(\exists -x \in V\,(x + (-x) = 0)\right)$

② V is closed under scalar multiplication

(i) $\alpha \cdot x \in V \quad , \quad \alpha \in \mathbb{F}, \; x \in V$

(ii) $\alpha(\beta x) = (\alpha\beta)x \quad , \quad \alpha, \beta \in \mathbb{F}, \; x \in V$

(iii) $\alpha(x+y) = \alpha x + \alpha y$

(iv) $1 \cdot x = x \qquad \forall x \in V$

(v) $(\alpha + \beta) \cdot x = \alpha x + \beta x$

Scalars have inverses, vector spaces don't.

<u>Observation 1</u>: The 0 vector is unique

Let $x + 0 = x \quad \forall x \in V$

Suppose $\exists\, \tilde{0} \in V$ s.t. $x + \tilde{0} = x \quad \forall x \in V$

Then, $0 + \tilde{0} = 0 = \tilde{0}$

<u>Observation 2</u>: Additive inverse is unique

Let $x \in V$ and suppose $\exists\, y \in V$ s.t.

$$x + y = 0$$

$$\Rightarrow \quad (-x) + x + y = -x \quad \Rightarrow \quad y = -x$$

Every element in a vector space is called a vector.

<u>Examples</u>

(1) $M_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$

In particular $M_{m\times 1}(\mathbb{F})$ and $M_{1\times n}(\mathbb{F})$ are vector spaces over $\mathbb{F}$

$$+ \rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} =$$

$$\cdot \rightarrow \quad [\alpha \cdot A]_{ij} = \alpha\,[A]_{ij}$$

(2) Co-ordinate Space over $\mathbb{R}$

$$\mathbb{R}^n := \{ v = (v_1, v_2, \ldots, v_n) \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

Define $v + w$ as

$$(v_1, v_2, \ldots, v_n) + (w_1, \ldots, w_n)$$

$$:= (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

set
+ relations b/w elements

structures

- field, vector space, groups
- category theory

For $\alpha \in \mathbb{R}$, define $\alpha \cdot v = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n)$

then $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

(3) Co-ordinate space over $\mathbb{C}$

The complex co-ordinate space $\mathbb{C}^n$ is also a vector space over $\mathbb{R}^1$

over $\mathbb{C}$ $\Rightarrow$ over $\mathbb{R}$

over $\mathbb{R}$ with usual multiplication $\not\Rightarrow$ over $\mathbb{C}$

$\mathbb{R}^2$ is a vector space over $\mathbb{R}$, not $\mathbb{C}$

$$i \cdot (1, 0) = (i, 0)$$

(4) Polynomial functions

Let $\mathcal{P}_n(\mathbb{F})$ be the set of all polynomials with degree at most n, with coefficients in $\mathbb{F}$

$$\mathcal{P}_n(\mathbb{F}) = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_1, a_2, \ldots, a_n \in \mathbb{F} \}$$

$\mathcal{P}_n(\mathbb{F})$ is a vector space over $\mathbb{F}$.

$$a_0 + a_1 x^1 + \cdots + a_n x^n$$

$$= \sum_{i=0} (a_i + b_i)\, x^i$$

$$+ \; b_0 + b_1 x^1 + \cdots + b_n x^n$$

and $$\alpha \cdot \left(\sum_{i=0}^{n} a_i x^i \right) = \sum_{i=0}^{n} \alpha a_i x^i$$

(5) Continuous functions

Let $\mathcal{C}[0, 1]$ denote the set of all continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$.

$\mathcal{C}[0, 1]$ is a vector space over $\mathbb{R}$, with respect to usual addition of functions and scalar multiplication

$$(\alpha \cdot f)(x) = \alpha \cdot f(x), \quad f \in \mathcal{C}[0,1], \; x \in [0,1]$$
$$\alpha \in \mathbb{F}$$

$$(f + g)(x) = f(x) + g(x)$$

(6) Line passing through origin $\quad \frac{x_1}{2} = \frac{x_2}{3} = \frac{x_3}{5}$

Consider the set of all triplets $(x_1, x_2, x_3) \in \mathbb{R}^3$ satisfying $\frac{x_1}{2} = \frac{x_2}{3} = \frac{x_3}{5}$

$$L_{(2,3,5)} := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \; \frac{x_1}{2} = \frac{x_2}{3} = \frac{x_3}{5} \right\}$$

$L(2,3,5)$ is a vector space over $\mathbb{R}$, with '+' and '·' taken from $\mathbb{R}^3$.

(7) $\mathbb{F}^X : \{f : X \to \mathbb{F}\}$

Let X be a non-empty set and let $\mathbb{F}^X$ denote the collection of all functions from X to $\mathbb{F}$.

Then, $\mathbb{F}^X$ is a vector space over $\mathbb{F}$ with usual function addition and scalar multiplication defined as

$$(\alpha \cdot f)(x) = \alpha \cdot f(x), \quad f \in \mathbb{F}^X,\ x \in X,\ \underbrace{\alpha \in \mathbb{F}}_{??}$$

$$(f+g)(x) = f(x) + g(x)$$

(8) Continuous functions that vanish at a fixed point in $[0,1]$