2024/07/30

Linear Algebra → 20th century

Gaussian elimination ← formalised 300BC ⇝ agricultural problem

17th - 18th century

planetary motion

Seki → Cramer → Euler → Gauss

1683 | 1750 | | 1811

3x3 det. | formalized | | elimination

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \text{System of equations}$$

eqⁿ

lines

coefficient

system of eqⁿˢ as an object.

$$\left[\begin{array}{cc:c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \vdots & \vdots & \vdots \end{array}\right] \text{array of numbers}$$

Sylvester → $\boxed{\text{Frobenius}}$ → Minkowski

"Matrix"

Vector Space & linear independence

— Google search —

arranging numbers in an array

Minkowski spacetime

↓

Einstein relativity

Peano

Vector Space

System of linear equations

Polynomial deg 1 = ~

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m$$

$x_i \longrightarrow$ variable

$a_{ij} \longrightarrow$ coefficients

$c_i \longrightarrow$ RHS

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

$m \times n$ $\quad n \times 1$ $\quad m \times 1$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

$$\left. a_{21} \quad a_{22} \right]_{2\times 2} \qquad \left. b_{12} \right]_{2\times 1} \longrightarrow 2\times 1$$

$$A = \left[A_1 \mid A_2 \mid \ldots \mid A_n \right]$$

columns

$$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \vdots \\ A_n \end{bmatrix}$$

$$A \cdot B$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Post-multiplication

$$A \cdot \left[B_1 \mid B_2 \right] = ?$$

$$\left[A \cdot B_1 \mid A \cdot B_2 \right]$$

$$= \left[A(B_1) \mid A(B_2) \right]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Pre-multiplication

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \cdot B = \begin{bmatrix} A_1(B) \\ A_2(B) \\ \vdots \\ A_n(B) \end{bmatrix}$$

Properties

(1) In general, $AB \neq BA$

(2) Associativity $A(BC) = (AB)C$

(3) Distributivity $\underbrace{A(B+C)}_{\text{valid orders}} = AB + AC$

(4) Scalar multiplication

$$\alpha \cdot \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ & \vdots & \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{bmatrix}$$

There exists a $n \times n$ matrix I, such that for any $n \times n$ matrix A

$$A \cdot I = I \cdot A = A$$

I is called the identity matrix

no other matrix that satisfies the above property

$$I = \begin{bmatrix} 1 & \cdots & & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Zero matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}_{n \times n} + X = X$$

Let A be a $n \times n$ matrix, then A is said to be invertible if there is a matrix U s.t.

$$UA = AU = I$$

U is denoted as A^{-1}.

Determinants

① $n \times n$ matrix $\longrightarrow$ plot in $\mathbb{R}_n$ $\longrightarrow$ oriented volume

② co-factors

Let A be $n \times n$ matrix of real or complex numbers

The determinant of A is

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} M_{ij}$$

M_{ij} is the minor of i^{th} row and j^{th} column.

- [ ] 1 and 2 same [3b1b playlist probably has the answer]
- [ ] det ab = det a . det b

field

Notation : $M_n(\mathbb{F})$ is the collection of all $n \times n$ matrices with entries in $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$M_n(\mathbb{R}) \qquad M_n(\mathbb{C})$$

$$M_{n \times m}(\mathbb{R})$$

* $\det(I) = 1$
* $\det(A + B) \neq \det(A) + \det(B)$

$$\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) \neq \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

* Let $A \in M_n(\mathbb{F})$ be invertible

$$\det(I) = \det(A^{-1}A)$$

$$\Rightarrow \quad 1 = \det(A^{-1})\det(A)$$

If A is invertible, $\det(A) \neq 0$

$$\Rightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)}$$

$\det(A) \cdot \det(B) = 1$ is a necessary but not sufficient condition for B to be A^{-1}

Next : Cramer's rule

BOOKS : K. Hoffman & K. Kunze : Linear Algebra

R. Rao & P. Bhimashankaran : Linear Algebra

Carl. D. Meyer : Matrix Analysis and Application

S. Kumaresan : Linear Algebra

Gilbert Strang

Sheldon Axler : Linear Algebra done right

comic

2024/08/02

Form - 1

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m$$

Form - 2

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

rectangular column column

Form - 3

$$A \in M_{m\times n}(\mathbb{F}) \quad \text{and} \quad C \in M_{m\times 1}(\mathbb{F})$$

$$AX = C$$

Let $A \in \underset{n\times n}{M_n}(\mathbb{F})$ and $C \in M_{n\times 1}(\mathbb{F})$. Consider the system:

$$AX = C$$

Suppose A is an invertible matrix.

$\Updownarrow$ equivalent

Then, $\det(A) \neq 0$

Claim: The system (I) has a solution.

i.e., there exists $b = [b_1, \dots, b_n]^T$ such that

$$A(b) = C$$

$$A(x) = C \Rightarrow x = A^{-1}C \qquad x = A^{-1}C$$

$$A^{-1}A(x) = A^{-1}C$$

$$b_i = ?$$

<u>Theorem</u> (Cramer's Rule)

Let $A \in M_n(\mathbb{F})$ be invertible, and $c \in M_{n\times 1}(\mathbb{F})$ then there is a unique solution to the system

$$A(x) = c \longrightarrow (*)$$

Moreover,

$$x_k := \det(A)^{-1} \left(\det A_k(c)\right), \qquad 1 \leq k \leq n,$$

where $A_k(c)$ is a $n \times n$ matrix formed by replacing the k^{th} column of A with c.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$$

$$A_k(c) = \begin{bmatrix} a_{11} & \cdots & c_1 & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & & \vdots \end{bmatrix}$$

$$\begin{bmatrix} a_{n1} & \cdots & c_n & & a_{nn} \end{bmatrix}$$

Proof:

Let $b = [b_1 \dots b_n]^T$ be a sol^n of $(*)$

i.e., $A(b) = C$

Consider the matrix

$$I_k(b) = \begin{array}{c} \begin{array}{ccccc} & & k^{th} & & \end{array} \\ \begin{bmatrix} 1 & \cdots & b_1 & \cdots & 0 \\ 0 & & \vdots & & \vdots \\ \vdots & & & & \\ & & b_n & & 1 \end{bmatrix} \end{array}$$

Laplace expansion along column 1

$$\det(I_{(k)}(b)) = b_k$$

$$\begin{aligned} b_k &= \det(I_{(k)}(b)) \\ &= \det((A^{-1}A)\, I_{(k)}(b)) \\ &= \det(A^{-1}(A\, I_{(k)}(b))) \\ &= \det(A^{-1}) \cdot \det(A\, I_{(k)}(b)) \\ &= \det(A)^{-1} \cdot \det(A\, I_{(k)}(b)) \longrightarrow (**) \end{aligned}$$

Claim: $A\, I_{(k)}(b) = A_k(C)$ for b to be a solⁿ

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & b_1 & \cdots & 0 \\ 0 & 1 & & & & \vdots \\ 0 & \vdots & & \vdots & & \vdots \\ \vdots & & & & & \\ & & & b_n & & 1 \end{bmatrix}$$

$$= \begin{array}{c} \\ \end{array} \begin{array}{ccccccc} & & & k^{th} \text{ column} & & \\ \end{array}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{11}b_1 + \cdots + a_{1n}b_n & \cdots & a_{1n} \\ \vdots & & & \vdots & & \\ a_{n1} & a_{n2} & \cdots & \underbrace{a_{n1}b_1 + \cdots + a_{nn}b_n}_{\left.\begin{matrix} c_1 \\ \vdots \\ c_n \end{matrix}\right\} \to \text{ by def}^n} & \cdots & a_{nn} \end{bmatrix}$$

$$= A_{(k)}(c)$$

$A \cdot I_{(k)}(b)$

$$e_j = [0, 0, \ldots, \overset{j^{th}}{1}, 0 \ldots 0]^T$$

$$A \cdot [\, e_1 \mid e_2 \mid \cdots \mid b \mid \cdots \mid e_n \,]$$

$$= [\, A \cdot e_1 \mid A \cdot e_2 \mid \cdots \mid A(b) \mid \cdots \mid A \cdot e_n \,]$$

$A \cdot e_j$ gives you the j^{th} column of A.

Hence, $\quad A \cdot I_{(k)}(b) = A_k(c) \qquad (\because \; A(b) = c)$

$(**)$ holds true.

$$b_k = \det(A)^{-1} \cdot \det(A_k(c)), \quad 1 \leq k \leq n.$$

$\square$

<u>Corollary</u>:

Let A be an invertible matrix. Then,

$$A^{-1}_{ij} = \det(A)^{-1} \cdot \det(A_i(e_j))$$

(i,j) entry of A^{-1}

$x_k := \det(A)^{-1} \cdot \det(A_k(c))$

RHS: $\quad A X^{(j)} = e_j$

$$A \cdot [\, X^{(1)} \mid X^{(2)} \mid \cdots \mid X^{(n)} \,] = [\, e_1 \mid e_2 \mid \cdots \mid e_n \,]$$

$$= I$$

$\square$

<u>System of eq^n</u>

$A \in M_{m \times n}(\mathbb{F})$,

$c \in M_{m \times 1}(\mathbb{F})$

$AX = c$

If $\quad c = 0$

then $\quad A(X) = 0 \quad$ is called a homogeneous equation.

Gaussian elimination

remove as many variables as possible

Form - 1

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m$$

Elementary Operations

1. If we swap i^{th} and j^{th} eq^n, the solution does not change
2. If we multiply $\alpha \neq 0$ with any eq^n, the solution does not change.
3. If we replace j^{th} eq^n by ($\alpha \neq 0$ times j^{th} eq^n) + (i^{th} eq^n), the solution does not change.

$$a_{i1}x_1 + \cdots + a_{in}x_n = c_i$$

$$a_{j1}x_1 + \cdots + a_{jn}x_n = c_j$$

$$\alpha(a_{i1}x_1 + \cdots + a_{in}x_n) + (a_{i1}x_1 + \cdots a_{in}x_n) = \alpha c_i + c_i$$

Augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_n \end{array}\right]$$

1. Swapping rows of the above matrix.

2. Multiply $\alpha \neq 0$ to any row.

Let i^{th} row of the above matrix be R_i , $i = 1, \dots, m$

3. $R_i \leftarrow R_i + \alpha R_j$

stair

Echelon form

$$\left.\left[\begin{array}{cccccc|c} 0 & 1 & x & x & x & x & x \\ 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x \\ & \cdots & & & & x & x \end{array}\right]\right\} \text{easier to solve}$$

Next : Echelon forms & reduced form